INDEFINITE THETA SERIES OF SIGNATURE $(1,1)$ FROM THE POINT OF VIEW OF HOMOLOGICAL MIRROR SYMMETRY

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Abstract. We apply the homological mirror symmetry for elliptic curves to the study of indefinite theta series. We prove that every such series corresponding to a quadratic form of signature (1,1) can be expressed in terms of theta series associated with split quadratic forms and the usual theta series. We also show that indefinite theta series corresponding to univalued Massey products between line bundles on elliptic curve are modular.

Introduction

The classical theta series are the series of the form $\sum_{\mathbf{n}\in\Lambda}\exp(\pi i\tau Q(\mathbf{n})+2\pi i\mathbf{n}\cdot\mathbf{z})$ where Q is a positive definite integer-valued quadratic form on a lattice Λ, $\mathbf{z}\in\Lambda_{\mathbb{C}}$, $\mathbf{z}\cdot\mathbf{z}':=\frac{1}{2}(Q(\mathbf{z}+\mathbf{z}')-Q(\mathbf{z})-Q(\mathbf{z}'))$ is the symmetric pairing on $\Lambda_{\mathbb{C}}$ induced by Q, τ belongs to the upper half-plane $\mathfrak{H}$. It is well-known that they are Jacobi forms on $\mathbb{C}\times\mathfrak{H}$ of weight $\operatorname{rk}\Lambda/2$ (see [2]). Now assume that Λ is a rank 2 lattice equipped with a non-degenerate $\mathbb{Q}$-valued quadratic form Q of signature $(1,1)$. Let us fix an open cone $C\subset\Lambda_{\mathbb{R}}$ of the form $C=\{\mathbf{v}\in\Lambda_{\mathbb{R}}:\phi(\mathbf{v})\cdot\psi(\mathbf{v})>0\}$ for a pair of linear forms ϕ and ψ on $\Lambda_{\mathbb{R}}$ defined over $\mathbb{Q}$, such that $Q|_C>0$. Let $C=C^+\cup C^-$ be the decomposition of C into two connected components, $\operatorname{sign}:C\to\{\pm1\}$ be the corresponding sign function (which is equal to 1 on C^+ and to -1 on C^-). Also let $\alpha:\Lambda_{\mathbb{C}}\to\Lambda_{\mathbb{R}}$ be the map given by $\alpha(\mathbf{z})=\operatorname{Im}(\mathbf{z})/\operatorname{Im}(\tau)$. Then we define the *indefinite theta series*[1] associated with (Λ,Q,C) by the formula

$$\Theta_{\Lambda,Q,C}(\mathbf{z},\tau)=\sum_{\mathbf{n}\in\Lambda:\mathbf{n}+\alpha(\mathbf{z})\in C}\operatorname{sign}(\mathbf{n}+\alpha(\mathbf{z}))\exp(\pi i\tau Q(\mathbf{n})+2\pi i\mathbf{n}\cdot\mathbf{z}). \tag{0.1}$$

This is a holomorphic function of τ (in the upper half-plane) and of the second variable $\mathbf{z}\in\Lambda_{\mathbb{C}}$ which varies in the complement to $\alpha^{-1}(\partial C+\Lambda)$ where ∂C is the boundary of C.

Our interest in the functions $\Theta_{\Lambda,Q,C}(\mathbf{z},\tau)$ is motivated by the observation that for some special choices of Λ, Q, C, vectors $\mathbf{v},\mathbf{w}\in\Lambda_{\mathbb{Q}}$ and a rational number λ the function

$$\exp(\pi i\lambda\tau)\Theta_{\Lambda,Q,C}(\tau\mathbf{v}+\mathbf{w},\tau)$$

is a modular form of weight 1 for some congruenz-subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In particular, the indefinite theta series defined by Hecke in [8] and their generalizations

[1] The definition of indefinite theta series by Göttsche and Zagier in [6] differs slightly from ours in that they fix a connected component of the domain of definition of $\Theta_{\Lambda,Q,C}$. Also we allow Q to take rational values; however, rescaling τ and z one can always reduce to the case of integer-valued form Q.

to arbitrary integral lattices of signature $(1,1)$ can be written in this form. By definition these series are

$$\Theta^H_{\Lambda,Q;\mathbf{c}}(\tau)=\sum_{\mathbf{n}\in\Lambda+\mathbf{c}/G;Q(\mathbf{n})>0}\operatorname{sign}(\mathbf{n})\exp(\pi i\tau Q(\mathbf{n})),$$

where we assume that $Q(\mathbf{n})\in 2\mathbb{Z}$ for all $\mathbf{n}\in\Lambda$, $\mathbf{c}\in\Lambda^{\perp}=\{\mathbf{x}\in\Lambda_{\mathbb{Q}}:\mathbf{x}\cdot\Lambda\subset\mathbb{Z}\}$, the function $\operatorname{sign}(\mathbf{n})$ takes opposite values ± 1 on two components of the cone $Q>0$, G is the subgroup in the identity component of the automorphism group of Q consisting of elements preserving $\Lambda+\mathbf{c}$. It is easy to see that such a series can be rewritten in the above form for some rational cone C: for this one can use the decomposition of the set of lattice points in the cone $Q>0$ described in [20].

Another example of modular behaviour of indefinite theta series goes back to Kronecker: one should consider the quadratic form $(m,n)\mapsto mn$ on $\mathbb{Z}^2$ and the cone $C=\{(x,y):xy>0\}$ (see [19],[21]). The corresponding series is a meromorphic Jacobi form on $\mathbb{C}\times\mathbb{C}\times\mathfrak{H}$. This example is related to “Teilwerte” of Weierstrass zeta-function considered by Hecke in [8].

The main point we would like to make in this paper is that these examples provide an evidence for the following conjecture: *an indefinite theta series of signature* $(1,1)$ *is modular if and only if it corresponds to a universal univalued triple Massey product in the derived categories of coherent sheaves on elliptic curves.* This correspondence which is based on homological mirror symmetry for elliptic curves (proved for transversal products in [16]) leads to explicit rational expressions for modular indefinite theta series in terms of the usual theta functions.

When the form Q is a product of rational linear forms, the function $\Theta_{\Lambda,Q,C}(\mathbf{z},\tau)$ (restricted to a connected component of its domain of definition) can be expressed via the following bilateral basic hypergeometric series

$$\kappa(y,x;\tau)=\sum_{m\in\mathbb{Z}}\frac{\exp(\pi i\tau m^2+2\pi imx)}{\exp(2\pi im\tau)-\exp(2\pi iy)}. \tag{0.2}$$

This series was introduced by M. P. Appell in his work [1] on decomposition of elliptic functions of the third kind into simple elements (see also [14]).

As the first application of our techniques we show that all indefinite theta series associated with quadratic forms of signature (1,1) and rational cones can be expressed in terms of κ and the usual theta series. To formulate this more precisely let us call a meromorphic function $\phi(z,\tau)$ on $\mathbb{C}\times\mathfrak{H}$ *elliptic* if it can be expressed rationally over $\mathbb{C}$ in terms of functions of the form $\theta_c(az+b\tau,d\tau)$ and $\exp(\pi id\tau)$, where $a,b,c\in\mathbb{Q}$, $d\in\mathbb{Q}_{>0}$, $\theta_c(z,\tau)=\sum_{n\in\mathbb{Z}+c}\exp(\pi i\tau n^2+2\pi inz)$. Let us also denote

$$\kappa_c(y,x;\tau)=\sum_{m\in\mathbb{Z}+c}\frac{\exp(\pi i\tau m^2+2\pi imx)}{\exp(2\pi im\tau)-\exp(2\pi iy)},$$

where $c\in\mathbb{Q}$.

Theorem 1. *For every triple* (Λ,Q,C) *as above and every connected open subset* $U\subset\Lambda_{\mathbb{C}}\setminus\alpha^{-1}(\partial C+\Lambda)$ *there exist* $\mathbb{Q}$*-linear functionals* $(r,s;l_i,i=1,\dots,N)$ *on* $\Lambda_{\mathbb{Q}}$*, constants* $(a_i,b_i,c_i,d_i,e_i,i=1,\dots,N)$ *in* $\mathbb{Q}$*,* $f\in\mathbb{Q}_{>0}$*, and meromorphic elliptic functions* $(\phi_i,\psi_i,i=1,\dots,N)$*, such that*

$$\Theta_{\Lambda,Q,C}(\mathbf{z},\tau)=\sum_{i=1}^{N}\phi_i(r(\mathbf{z}))\psi_i(s(\mathbf{z}))\kappa_{e_i}(a_i\tau+b_i,l_i(\mathbf{z})+c_i\tau+d_i;f\tau)$$

for $\mathbf{z} \in U$, $\tau \in \mathfrak{H}$.

The proof uses the interpretation of indefinite theta series as components of triple Fukaya products on a symplectic torus and the A_∞-identity connecting m_2 and m_3. These products were first defined by Fukaya in [3], and in a slightly more general form by Kontsevich in [10]. Basically, the triple products we need correspond to configurations of four lines with rational slopes on $\mathbb{R}^2$: they are defined as sums of exponents of areas of the series of quadrangles attached to such a configuration. These series are always given by some indefinite theta series as above. It turns out that when two of the four lines are parallel (i.e. one has a *trapezoid* configuration) then the corresponding quadratic form splits over $\mathbb{Q}$. Now using A_∞-constraints one can express any triple Fukaya product on a torus in terms of triple products corresponding to trapezoid configurations, hence the above theorem.

The following theorem provides examples of modular indefinite theta series for which the above principle holds (in other words, the series in this theorem correspond to some univalued Massey products on elliptic curve). Let us introduce a special notation for the summation pattern used for indefinite theta series: for $S \subset \mathbb{Q}^2$ we denote

$$\sum_{(m,n)\in S}^{indef} a_{m,n} := \sum_{(m,n)\in S, m\geq 0, n\geq 0} a_{m,n} - \sum_{(m,n)\in S, m<0, n<0} a_{m,n}.$$

Theorem 2. *Let* a, b, c, p *be positive integers such that* $a|b$, $c|b$, $p|(b/a+1)$, $p|(b/c+1)$ *and* $D = b^2 - ac > 0$. *Let also* s_1, s_2 *be odd integers and let* r *be a residue in* $\mathbb{Z}/p\mathbb{Z}$. *Then the series*

$$q^{\frac{p^2 ac(2bs_1s_2 - as_1^2 - cs_2^2)}{8D}} \cdot \sum_{(m,n)\in\mathbb{Z}^2, m\equiv n\equiv r(p)}^{indef} (-1)^{\frac{n-m}{p}} q^{bmn + a\frac{m^2+mps_1}{2} + c\frac{n^2+nps_2}{2}}$$

is a (meromorphic at cusps) modular form of weight 1 *with respect to some congruenz-subgroup of* $\mathrm{SL}_2(\mathbb{Z})$. *More precisely, it can be written as a ratio* $(\sum_{i=1}^k P_i/Q_i)/(\sum_{j=1}^l R_j/S_j)$, *where* P_i, Q_i, R_j *and* S_j *are polynomials in theta functions (with characteristics) of degrees* $2(D-2)$, $4(D-2)$, $2(D-1)$ *and* $4(D-1)$ *respectively.*

In the particular case $a = c = p = 1$, $r = 0$, $b > 1$ we get the series

$$q^{\frac{2bs_1s_2 - s_1^2 - s_2^2}{8(b^2-1)}} \cdot \sum_{(m,n)\in\mathbb{Z}^2}^{indef} (-1)^{m+n} q^{bmn + \frac{m^2+ms_1}{2} + \frac{n^2+ns_2}{2}}$$

considered in [9]. These series (for various s_1, s_2) coincide with the string functions of highest weight modules over $A_1^{(1)}$ of level $b-1$, multiplied by η^3, where η is the Dedekind eta-function (see section 7.3 for details). As was shown in [9] they are equal to Hecke's indefinite theta series of certain quadratic modules. We generalize this observation in the following theorem.

Theorem 3. *The series considered in Theorem 2 is equal to*

$$N \cdot \Theta^H_{\Lambda,Q;\mathbf{c}}(p^2\tau)$$

for some non-zero integer N, *where*

$$\Lambda = \{(m,n) \in \mathbb{Z}^2 : n \equiv (\frac{b}{c} + 1)m \mod (2)\},$$

$$\frac{1}{2}Q(m,n) = cn^2 - \frac{D}{c}m^2,$$

$$\mathbf{c} = (\frac{r}{p} + \frac{acs}{2D}, \frac{1}{2}).$$

Here $s = \frac{b}{a}s_2 - s_1$, so it can be an arbitrary integer such that $s \equiv \frac{b}{a} + 1 \mod (2)$.

Theorem 2 is a consequence of Theorem 4 below. We use the following notation: for a subgroup $I \subset \mathbb{Z}$ and an element $c \in \mathbb{Q}/I$ we denote

$$\theta_{I,c}(z,\tau) = \sum_{m \in c+I} \exp(\pi i \tau m^2 + 2\pi i m z).$$

Sometimes we abbreviate $\theta_{I,0}$ to θ_I and $\theta_{\mathbb{Z},c}$ to θ_c.

Theorem 4. *Let d_0, d_1, d_2 and d be positive integers satisfying $d_0 + d = d_1 + d_2$, $d_1 < d$, $d_2 < d$. Let Q be the following quadratic form:*

$$Q(m,n) = \frac{1}{d}(d_1(d-d_1)m^2 + 2d_1 d_2 mn + d_2(d-d_2)n^2).$$

Fix τ in the upper half-plane and let (x_1, x_2) be a pair of complex numbers satisfying

$$x_1 + x_2 + \frac{1}{2d_1} - \frac{1}{2d_2} \notin \frac{1}{N}\mathbb{Z} + \mathbb{Z}\tau, \tag{0.3}$$

where N is the least common multiple of d_1 and d_2. For a collection of complex numbers $(c_k, k \in \mathbb{Z}/d\mathbb{Z})$ and an integer l, $0 \le l < d_0$, let us consider the series

$$F_l = - \sum_{(m,n) \in \mathbb{Z}^2}^{indef} c_{d_1 m - d_2 n + l} a_{m,n,l},$$

where

$$a_{m,n,l} = \exp(\pi i \tau Q(m + \frac{l}{d_0}, n - \frac{l}{d_0}) + 2\pi i [d_1 x_1 (m + \frac{l}{d_0}) + d_2 x_2 (n - \frac{l}{d_0})]).$$

Assume that we have

$$\sum_{n \in \mathbb{Z}} c_{d_1 m_0 - d_2 n + l} a_{m_0,n,l} = \sum_{m \in \mathbb{Z}} c_{d_1 m - d_2 n_0 + l} a_{m,n_0,l} = 0 \tag{0.4}$$

for all $m_0, n_0 \in \mathbb{Z}$ and all l. Then F_l are uniquely determined from the linear system of equations

$$\sum_{l \in \mathbb{Z}/d_0\mathbb{Z}} D_{k,l} F_l = c_k$$

for $k \in \mathbb{Z}/d\mathbb{Z}$, where

$$D_{k,l} = \frac{1}{d_1 i \eta^3(d_1 \tau)} \times$$

$$\sum_{a \in \mathbb{Z}/d_1\mathbb{Z}} (-1)^a \frac{\theta_{d_0\mathbb{Z}, l + \frac{d_0}{2}}(\frac{(d_2 - d)x_1 + d_2 x_2}{d_0} + \frac{2a+1}{2d_1}, \frac{\tau}{d_0}) \theta_{d\mathbb{Z}, -k + \frac{d}{2}}(x_1 + \frac{2a+1}{2d_1}, \frac{\tau}{d})}{\theta_{d_2\mathbb{Z}, \frac{d_2}{2}}(x_1 + x_2 + \frac{2a+1}{2d_1}, \frac{\tau}{d_2})},$$

$\eta(\tau) = q^{1/24} \cdot \prod_{n \ge 1}(1 - q^n)$ is the Dedekind eta-function.

The proof of this theorem is also based on the interpretation of indefinite theta series as matrix coefficients of triple Fukaya products on a torus. We use the homological mirror symmetry for elliptic curve (see [16]) to relate these products to Massey products in the derived category of an elliptic curve.

In the case $d_0 = 1$ Theorem 4 gives an explicit formula for the series F_0. In particular, we obtain the following interesting identities between q-series.

Corollary 5. *One has*

$$q^{\frac{1}{12}} \sum_{(m,n)\in\mathbb{Z}^2}^{indef} (-1)^{m+n} q^{2mn+\frac{m^2+m}{2}+\frac{n^2+n}{2}} = \eta(\tau)^2, \tag{0.5}$$

$$\sum\nolimits_{(m,n)\in\mathbb{Z}^2, m\equiv n+1(2)}^{indef} (-1)^{\frac{m+n-1}{2}} q^{\frac{m^2+6mn+3n^2}{2}} = \frac{\eta^3(2\tau)\theta_{\frac{1}{2}}(\frac{1}{4},3\tau)}{\theta(\frac{1}{2},4\tau)\theta_{\frac{1}{2}}(\frac{1}{4},\tau)} = q^{\frac{1}{2}} \cdot \prod\nolimits_{n\geq 1}(1+q^n)(1-q^{2n})(1-q^{3n})(1+q^{6n}), \tag{0.6}$$

$$\sum\nolimits_{m\in\mathbb{Z}+\frac{1}{2}, n\in\mathbb{Z}}^{indef} (-1)^{m+n-\frac{1}{2}} q^{\frac{m^2+6mn+3n^2}{2}} = \frac{\eta^3(2\tau)\theta(\frac{1}{4},3\tau)}{\theta_{\frac{1}{4}}(0,4\tau)\theta(\frac{1}{4},\tau)} = q^{\frac{1}{8}} \cdot \prod\nolimits_{n\geq 1}(1-q^n)(1+q^{2n})(1-q^{3n})(1+q^{6n-3}), \tag{0.7}$$

$$\sum\nolimits_{m\in\mathbb{Z}, n\in\mathbb{Z}+\frac{1}{2}}^{indef} (-1)^{m+n-\frac{1}{2}} q^{\frac{m^2+6mn+3n^2}{2}} = \frac{\eta^3(2\tau)\theta_{\frac{1}{2}}(\frac{1}{4},3\tau)}{\theta_{\frac{1}{4}}(0,4\tau)\theta_{\frac{1}{2}}(\frac{1}{4},\tau)} = q^{\frac{3}{8}} \cdot \prod\nolimits_{n\geq 1}(1-q^n)(1+q^{2n-1})(1-q^{3n})(1+q^{6n}). \tag{0.8}$$

Identity (0.5) was obtained in [9] (formula (5.19)) by representation-theoretic means. Three other identities above seem to be new.

Note that since indefinite theta series are given by alternating sums, the important problem (raised already by Hecke in [8]) is to determine exactly which of them vanish identically. There is a necessary condition (cf. Satz 2 in [8]): if $\Theta^H_{\Lambda,Q;\mathbf{c}} \neq 0$ then every automorphism of Q preserving $\Lambda + \mathbf{c}$ should preserve each component of the cone $Q > 0$. We will prove the following non-vanishing result.

Theorem 6. *In the notations of Theorem 2 consider the series*

$$f_{s_1,s_2} = q^{\frac{p^2ac(2bs_1s_2-as_1^2-cs_2^2)}{8D}} \cdot \sum_{(m,n)\in\mathbb{Z}^2, m\equiv n(p)}^{indef} (-1)^{\frac{n-m}{p}} \zeta_p^{rm} q^{bmn+a\frac{m^2+mps_1}{2}+c\frac{n^2+nps_2}{2}},$$

where ζ_p is the primitive root of unity of order p. Let us denote by h the greatest common divisor of $b/a+1$, $b/c+1$. Assume that either

$$\frac{s_1+s_2}{2} \notin \frac{h}{p}\mathbb{Z}$$

or

$$r \notin \frac{(h+p)(2b+a+c)}{2hb} + \frac{p}{b}(a\mathbb{Z}+c\mathbb{Z}).$$

Then there exist integers l_1 and l_2 such that

$$f_{s_1+2\frac{h}{p}l_1, s_2+2\frac{h}{p}l_2} \neq 0$$

Together with Theorem 3 this leads to the following

Corollary 7. *In the notations of Theorems 3 and 6 consider the collection of characteristics*

$$\mathbf{c}(t) = (\frac{(b+a)c}{2D} + \frac{act}{D}, \frac{1}{2})$$

where $t \in \mathbb{Z}$. Then for every non-zero residue $\overline{t}$ modulo $\frac{h}{p}$ there exists $t \equiv \overline{t}(\frac{h}{p})$ such that $\Theta^H_{\Lambda,Q;\mathbf{c}(t)} \neq 0$.

Using similar techniques we will show in Theorem 9 that certain functions of the form

$$\sum_i c_i(\mathbf{z})\Theta_{\Lambda,Q,C}(\mathbf{z} + \mathbf{v}_i\tau + \mathbf{w}_i, \tau)$$

are meromorphic Jacobi forms (for a congruenz-subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and with respect to some quadratic form Q' on a sublattice of Λ) in the sense of the definition given by L. Göttsche and D. Zagier in [6]. Here $(\mathbf{v}_i, \mathbf{w}_i)$ is a collection of vectors in $\Lambda_{\mathbb{Q}}$, the coefficients $c_i(\mathbf{z})$ are products of elliptic functions of some linear functionals of $\mathbf{z}$. In the case of a split form Q we obtain the following result.

Theorem 8. *The series*

$$u^{\frac{s}{2a}} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n^2+sn}{2}}}{1 - q^{an}u}, \tag{0.9}$$

where a is a positive integer and s is an odd integer, defines a meromorphic Jacobi form (here we use multiplicative variables $q = \exp(2\pi i\tau)$ and $u = \exp(2\pi iz)$).

In fact, Theorem 4 gives an explicit (although complicated) rational expression for the series (0.9) in terms of theta functions. In the case $a = 1$ such an expression is well known (see [18], Section 486, or [9] (5.26), or [14]). In the case $a = 2$ this expression is given by formula (7.7).

Considering higher Fukaya products m_k with $k \geq 4$ on a symplectic torus one still gets some indefinite theta series corresponding to lattices with quadratic forms of signature $(1, k-2)$. However, quadratic forms associated with configurations of $k+1$ lines depend on k parameters while a general quadratic form on a lattice of rank $k-1$ has $(k-1)k/2$ coefficients. Hence, for $k \geq 4$ not every indefinite series associated with a quadratic form of signature $(1, k-2)$ comes from a Fukaya product.

Here is the plan of the paper. In section 1 we explain the relation between indefinite theta series of signature $(1,1)$ and the Appell's function (0.2). Section 2 contains the definition of Fukaya products on a symplectic torus and the computation of double and triple Fukaya products in terms of the usual theta functions and indefinite theta series respectively. In section 3 we prove an auxiliary surjectivity result about the products m_2 in the Fukaya category of a symplectic torus. In section 4 we prove Theorem 1 and in section 5 we illustrate it by an explicit example. Section 6 is devoted to the definition and computation of Massey products of morphisms between line bundles on elliptic curve. In section 7 we give examples of modular indefinite theta series, proving in particular Theorems 2, 3, 4, 6, 8 and Corollary 5.

Acknowledgment. During the preparation of this paper I benefited from conversations with B. Gross, M. Kontsevich and D. Zagier. I am grateful to V. Kac for the reference to Hecke's works and to [9]. Part of this paper was written during my

visit to Max-Planck-Institut für Mathematik. I'd like to thank the Institute for its hospitality. This work was partially supported by the NSF grant.

1. Indefinite theta series of signature $(1,1)$

Let Λ be a rank 2 lattice equipped with a $\mathbb{Q}$-valued quadratic form Q, τ be an element in the upper-half plane. We assume that Q has signature $(1,1)$ and fix a rational open cone $C \in \Lambda_{\mathbb{R}}$ such that $Q|_C > 0$. We will use the following notation for indefinite theta series with characteristics associated with (Λ, Q, C): for an element $\mathbf{c} \in \Lambda_{\mathbb{Q}}/\Lambda$ we set

$$\Theta_{\Lambda,Q,C;\mathbf{c}}(\mathbf{z},\tau) = \sum_{\mathbf{n}\in\mathbf{c}+\Lambda:\mathbf{n}+\alpha(\mathbf{z})\in C} \operatorname{sign}(\mathbf{n}+\alpha(\mathbf{z}))\exp(\pi i\tau Q(\mathbf{n}) + 2\pi i\mathbf{n}\cdot\mathbf{z}) \tag{1.1}$$

In other words, we have

$$\Theta_{\Lambda,Q,C;\mathbf{c}}(\mathbf{z},\tau) = \exp(\pi i\tau Q(\mathbf{c}) + 2\pi i\mathbf{c}\cdot\mathbf{z})\Theta_{\Lambda,Q,C}(\mathbf{z}+\tau\mathbf{c},\tau).$$

The following identities follow immediately from the definition:

$$\Theta_{N\Lambda,Q,C,\mathbf{c}}(\mathbf{z},\tau) = \Theta_{\Lambda,Q,C,\frac{\mathbf{c}}{N}}(N\mathbf{z}, N^2\tau),$$

$$\Theta_{\Lambda,NQ,C,\mathbf{c}}(\mathbf{z},\tau) = \Theta_{\Lambda,Q,C,\mathbf{c}}(N\mathbf{z}, N\tau)$$

where $N > 0$ in an integer,

$$\Theta_{\Lambda,Q,C,\mathbf{c}}(\mathbf{z},\tau) = \sum_{\mathbf{n}\in\Lambda/\Lambda'} \Theta_{\Lambda',Q,C;\mathbf{c}+\mathbf{n}}(z,\tau)$$

for any sublattice $\Lambda' \subset \Lambda$. Since $N\Lambda \subset \Lambda'$ for some N we can also use these formulas to express $\Theta_{\Lambda',Q,C}$ in terms of $\Theta_{\Lambda,Q,C}$.

On the other hand, since the cone C is rational we can choose coordinates in such a way that $C = \{(x_1,x_2) \in \mathbb{R}^2 : x_1x_2 > 0\}$, $x_i > 0$ in C^+ and Λ is a lattice in $\mathbb{R}^2$ commensurable with $\mathbb{Z}^2$. Now the condition $Q|_C > 0$ and the requirement that the signature of Q is $(1,1)$ mean that $Q(x_1,x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ where $a_{ii} \geq 0$ and $D = a_{12}^2 - a_{11}a_{22} > 0$.

Now let us consider the case when Q splits into a product of linear forms over $\mathbb{Q}$. Then by additivity of Θ in C it suffices to consider the case when Q vanishes on one of the lines forming the boundary of C. Then we can choose coordinates in such a way that $C = \{(x_1,x_2) \in \mathbb{R}^2 : x_1x_2 > 0\}$, $x_i > 0$ in C^+, $Q(n_1,n_2) = an_1(n_1+2n_2)$ for some $a \in \mathbb{Q}$, Λ is a lattice commensurable with $\mathbb{Z}^2$. Rescaling τ and $\mathbf{z}$ (rationally) we can assume that $a = 1$. Also it suffices to consider the case $\Lambda = \mathbb{Z}^2$. Then for any $\mathbf{c} = (c_1,c_2)$ we have

$$\Theta_{\mathbb{Z}^2,Q,C,\mathbf{c}}(\mathbf{z},\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^2+\mathbf{c},(n_1+\alpha(z_1))(n_2+\alpha(z_2))>0} \operatorname{sign}(n_1+\alpha(z_1)) \times$$
$$\exp(\pi i\tau n_1(n_1+2n_2) + 2\pi i(n_1z_2+n_2z_1) + 2\pi i n_1 z_1).$$

We can split this sum in two pieces and sum the geometric progression in n_2 in each of them:

$$\Theta_{\mathbb{Z}^2,Q,C,\mathbf{c}}(\mathbf{z},\tau) =$$
$$\sum_{n_1\in\mathbb{Z}+c_1, n_1+\alpha(z_1)>0} \exp(\pi i\tau n_1^2 + 2\pi i n_1(z_1+z_2)) \sum_{n_2\in\mathbb{Z}_{\geq 0}+n_2^0} \exp(2\pi i n_2(\tau n_1 + z_1)) -$$
$$\sum_{n_1\in\mathbb{Z}+c_1, n_1+\alpha(z_1)<0} \exp(\pi i\tau n_1^2 + 2\pi i n_1(z_1+z_2)) \sum_{n_2\in\mathbb{Z}_{\leq 0}+n_2^0-1} \exp(2\pi i n_2(\tau n_1 + z_1)) =$$
$$\sum_{n_1\in\mathbb{Z}+c_1, n_1+\alpha(z_1)>0} \frac{\exp(\pi i\tau n_1^2 + 2\pi i n_1(z_1+z_2) + 2\pi i n_2^0(\tau n_1+z_1))}{1-\exp(2\pi i(\tau n_1+z_1))} -$$
$$\sum_{n_1\in\mathbb{Z}+c_1, n_1+\alpha(z_1)<0} \frac{\exp(\pi i\tau n_1^2 + 2\pi i n_1(z_1+z_2) + 2\pi i(n_2^0-1)(\tau n_1+z_1))}{1-\exp(-2\pi i(\tau n_1+z_1))} =$$
$$\sum_{n_1\in\mathbb{Z}+c_1} \frac{\exp(\pi i\tau n_1^2 + 2\pi i n_1(z_1+z_2) + 2\pi i n_2^0(\tau n_1+z_1))}{1-\exp(2\pi i(\tau n_1+z_1))}$$

where n_2^0 is the minimal $n_2 \in \mathbb{Z}+c_2$ such that $n_2+\alpha(z_2)>0$. Hence, we derive the following formula:

$$\Theta_{\mathbb{Z}^2,Q,C,\mathbf{c}}(\mathbf{z},\tau) = \exp(2\pi i n_2^0 z_1)\kappa_{c_1}(z_1,(1-n_2^0)\tau - z_1 - z_2;\tau). \tag{1.2}$$

2. Fukaya category of a torus

2.1. **Definition.** Let us recall the definition of the Fukaya A_∞-category of the torus $\mathbb{R}^2/\mathbb{Z}^2$ with the (complexified) symplectic form $-2\pi i\tau dx\wedge dy$ where τ is an element of the upper half-plane (for more details see [16]). More precisely, it is not quite an A_∞-category since morphisms are only defined for transversal configurations of objects, however, the axiomatics can be changed appropriately (see [11], sec. 4.3) Also, we will need only the subcategory $\mathcal{F}_s$ which is described as follows. The objects of $\mathcal{F}_s$ are pairs (L,t) where $L\subset\mathbb{R}^2$ is a non-vertical line with rational slope considered modulo translations by $\mathbb{Z}^2$, t is a real number. Morphisms between two such objects (L_1,t_1) and (L_2,t_2) are defined only if $L_1 \neq L_2 \mod \mathbb{Z}^2$. In this case $\mathrm{Hom}((L_1,t_1),(L_2,t_2)) = \mathrm{Hom}(L_1,L_2)$ is a $\mathbb{C}$-vector space with the basis $[P]$ enumerated by points $P\in(L_1+\mathbb{Z}^2)\cap(L_2+\mathbb{Z}^2)$ modulo $\mathbb{Z}^2$ (the numbers t_i will play a role only in the definition of compositions). Let λ_i be the slope of the line L_i $(i=1,2)$. Then $\mathrm{Hom}(L_1,L_2)\neq 0$ only if $\lambda_1\neq\lambda_2$. This space has grading 0 if $\lambda_1<\lambda_2$ and grading 1 if $\lambda_1>\lambda_2$. By definition the differential m_1 is zero. The compositions m_k for $k\geq 2$ are (partially) defined as follows. Let $L_0,L_1,\dots,L_k$ be the set of lines in $\mathbb{R}^2$ with slopes $\lambda_0,\lambda_1,\dots,\lambda_k$. Assume that the images of L_i in $\mathbb{R}^2/\mathbb{Z}^2$ form a *transversal configuration*, i.e., no three of them intersect in one point. For every $i=0,\dots,k-1$ let d_i be the grading of $\mathrm{Hom}(L_i,L_{i+1})$. The composition

$$m_k : \mathrm{Hom}(L_0,L_1)\otimes\ldots\otimes\mathrm{Hom}(L_{k-1},L_k) \to \mathrm{Hom}(L_0,L_k)$$

is non-zero only if $\sum_{i=0}^{k-1} d_i - k + 2$ is equal to the degree of $\mathrm{Hom}(L_0, L_k)$. Let $P_{i,i+1}$ be some intersection points of L_i and L_{i+1} modulo $\mathbb{Z}^2$. Then

$$m_k([P_{0,1}], [P_{1,2}], \dots, [P_{k-1,k}]) =$$
$$\sum_{P_{0,k},\Delta} \pm \exp\left(2\pi i\tau \cdot \int_\Delta dx \wedge dy + 2\pi i \sum_{j\in\mathbb{Z}/(k+1)\mathbb{Z}} (x(p_j) - x(p_{j-1}))t_j\right) [P_{0,k}]$$

where the sum is taken over points of intersections $P_{0,k}$ of L_0 with L_k modulo $\mathbb{Z}^2$ and over all $(k+1)$-gons Δ (considered up to traslation by $\mathbb{Z}^2$) with vertices $p_i \equiv P_{i,i+1} \mod \mathbb{Z}^2$, $i \in \mathbb{Z}/(k+1)\mathbb{Z}$, such that the edge $[p_{i-1}, p_i]$ belongs to $L_i + \mathbb{Z}^2$. We also require that the path formed by the edges $[p_0, p_1], [p_1, p_2], \dots, [p_k, p_0]$ goes in the clockwise direction. The sign in the RHS is "plus" if k is even and is equal to the sign of $x(p_0) - x(p_k)$ if k is odd.

The A_∞-constraint we are going to use is

$$m_3(m_2(a_1, a_2), a_3, a_4) - m_3(a_1, m_2(a_2, a_3), a_4) + m_3(a_1, a_2, m_2(a_3, a_4)) =$$
$$= m_2(m_3(a_1, a_2, a_3), a_4) + (-1)^{\deg(a_1)} m_2(a_1, m_3(a_2, a_3, a_4)), \quad (2.1)$$

where $a_1, \dots, a_4$ are composable morphisms between 5 objects in $\mathcal{F}_s$ forming a transversal configuration. Below we will often abbreviate $m_2(a, b)$ to ab.

2.2. **Double products and vector bundles on elliptic curves.** Since $m_1 = 0$ the composition m_2 is associative, so we can consider the category $\mathcal{F}_s$ with m_2 as a usual category. It was shown in [17] that the obtained category is equivalent to the category of stable vector bundles on the elliptic curve $E = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$, where morphisms between vector bundles V_1 and V_2 are elements of the graded vector space $\oplus_i \mathrm{Ext}^i(V_1, V_2)$ (in fact, we showed in [17] how to extend this equivalence to the whole derived category of coherent sheaves on elliptic curve, but we don't need this extension here). The construction of this equivalence (which we recall below) is based on the observation due to M. Kontsevich that the Fukaya product m_2 on a torus is given essentially by theta functions. Here is a more precise statement.

For a pair (λ, y) where $\lambda \in \mathbb{Q}$, $y \in \mathbb{R}$, let us denote by $L(\lambda, y)$ the line in $\mathbb{R}^2$ given by

$$L(\lambda, y) = \{(t, \lambda t - y), t \in \mathbb{R}\}.$$

Now let $L_i = L(\lambda_i, y_i)$, $i = 0, 1, 2$, be lines in $\mathbb{R}^2$ with distinct slopes λ_i. Let us denote

$$y_{ij} = \frac{y_j - y_i}{\lambda_j - \lambda_i},$$
$$y'_{ij} = \frac{\lambda_i y_j - \lambda_j y_i}{\lambda_j - \lambda_i}.$$

The lines L_i and L_j intersect at the point

$$P_{ij}(y_i, y_j) = (y_{ij}, y'_{ij}).$$

Note that if we change y_j by $y_j + m\lambda_j + n$ where $m, n \in \mathbb{Z}$, the new line $L(\lambda_j, y_j + m\lambda_j + n)$ is a shift of L_j by an integer vector. Thus, the new point of intersection $P_{ij}(y_i, y_j + m\lambda_j + n)$ still belongs to $L_i \cap (L_j + \mathbb{Z}^2)$. One has $P_{ij}(y_i, y_j + m\lambda_j + n) \equiv P_{ij}(y_i, y_j) \mod \mathbb{Z}^2$ if and only if $(m, n) \in \Lambda(\lambda_i, \lambda_j)$ where

$$\Lambda(\lambda_i, \lambda_j) = \{(m, n) \in \mathbb{Z}^2 : \frac{m\lambda_j + n}{\lambda_j - \lambda_i} \in I_{\lambda_i}\}$$

where for every $\lambda \in \mathbb{Q}$ we denote

$$I_\lambda = \{n \in \mathbb{Z} : \lambda n \in \mathbb{Z}\}. \tag{2.2}$$

Thus, we have the following basis in $\operatorname{Hom}(L_i, L_j)$:

$$[P_{ij}(y_i, y_j + m\lambda_j + n)], (m, n) \in \mathbb{Z}^2/\Lambda(\lambda_i, \lambda_j).$$

Instead of shifting y_j we could also shift y_i and get a different indexing of intersection points modulo $\mathbb{Z}^2$. However, this indexing is related to the previous one by the formula

$$P_{ij}(y_i - m\lambda_i - n, y_j) = P_{ij}(y_i, y_j + m\lambda_j + n).$$

Note also that we have $\Lambda(\lambda_i, \lambda_j) = \Lambda(\lambda_j, \lambda_i)$, so changing the order of lines we would get essentially the same indexing.

Assume that $\deg \operatorname{Hom}(L_0, L_1) + \deg \operatorname{Hom}(L_1, L_2) = \deg \operatorname{Hom}(L_0, L_2)$. Let t_i, $i = 0, 1, 2$, be some real numbers. Then we can consider objects (L_i, t_i) in Fukaya category. An easy computation shows that

$$m_2([P_{01}(y_0, y_1)], [P_{12}(y_1, y_2)]) =$$
$$\sum_{n \in I_{\lambda_1}} \exp(\pi i \tau p(v_1 + n)^2 - 2\pi i p(v_1 + n)w_1)[P_{02}(y_0, y_2 + n\lambda_2 - n\lambda_1)] \tag{2.3}$$

where

$$p = p(\lambda_0, \lambda_1, \lambda_2) = \frac{(\lambda_2 - \lambda_1)(\lambda_1 - \lambda_0)}{(\lambda_2 - \lambda_0)},$$

$v_1 = y_{12} - y_{01}$, $w_1 = t_{12} - t_{01}$,

$$t_{ij} = \frac{t_j - t_i}{\lambda_j - \lambda_i},$$

I_{λ_1} is defined by (2.2). Note that the class of the point $[P_{02}(y_0, y_2 + n\lambda_2 - n\lambda_1)]$ modulo $\mathbb{Z}^2$ depends only on the class of n modulo the following subgroup

$$I_{\lambda_0,\lambda_1,\lambda_2} = I_{\lambda_1} \cap \frac{\lambda_2 - \lambda_0}{\lambda_2 - \lambda_1} I_{\lambda_0}.$$

The matrix coefficients of the above product are given by values of elliptic functions at $(\tau v_1 - w_1, \tau)$ times the non-holomorphic factor $\exp(\pi i \tau p v_1^2 - 2\pi i p v_1 w_1)$. One can get rid of this factor by rescaling the bases in $\operatorname{Hom}(L_i, L_j)$ appropriately. Namely, we set

$$e_{ij}(m, n) = e_{y_i,y_j}(m, n) = \exp(\pi i \tau(\lambda_i - \lambda_j) y_{ij}^2 - 2\pi i (t_i - t_j) y_{ij})[P_{ij}(y_i, y_j + m\lambda_j + n)] \tag{2.4}$$

where $(m, n) \in \mathbb{Z}^2/\Lambda(\lambda_i, \lambda_j)$. Then the above formula is equivalent to

$$m_2(e_{01}(0, 0), e_{12}(0, 0)) = \sum_{n \in I_{\lambda_1}/I_{\lambda_0,\lambda_1,\lambda_2}} \theta_{I_{\lambda_0,\lambda_1,\lambda_2},n}(p(v_1\tau - w_1), p\tau) e_{02}(n, -n\lambda_1)$$

where we use the notation $\theta_{I,c}$ from the introduction. Changing y_i's appropriately in the formula (2.3) we derive a more general formula

$$m_2(e_{01}(a, b), e_{12}(c, d)) =$$
$$\sum_{n \in I_{\lambda_1}/I_{\lambda_0,\lambda_1,\lambda_2}} \theta_{I_{\lambda_0,\lambda_1,\lambda_2},u+n}(p(v_1\tau - w_1), p\tau) e_{02}(a + c + n, b + d - n\lambda_1) \tag{2.5}$$

where

$$u = \frac{c\lambda_2 + d}{\lambda_2 - \lambda_1} - \frac{a\lambda_0 + b}{\lambda_1 - \lambda_0}.$$

The corresponding coefficients will be holomorphic in $v_1\tau - w_1$. The associativity condition for m_2 is equivalent to the classical addition formulas for theta-functions.

The equivalence with the category of stable bundles on elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is constructed in [17] as follows. First let us consider the subcategory in $\mathcal{F}_s$ formed by lines with integer slopes. To an object of this subcategory $(L(\lambda, y), t)$ where $\lambda \in \mathbb{Z}$, $y, t \in \mathbb{R}$, we associate the line bundle $t^*_{y\tau - t}\mathcal{L} \otimes \mathcal{L}^{\otimes(\lambda-1)}$ where $\mathcal{L} = \mathcal{L}_\tau$ is the line bundle of degree 1 on E_τ such that $\theta(z, \tau)$ is a holomorphic section of $\mathcal{L}_\tau$, $t_z : E_\tau \to E_\tau$ denotes the translation by z. Assume that we have two such objects (L_i, t_i), $i = 1, 2$, where $L_i = L(\lambda_i, y_i)$, $\lambda_i \in \mathbb{Z}$, $\lambda_1 < \lambda_2$. Then we identify $\mathrm{Hom}(L_1, L_2)$ with the space of morphisms between the corresponding line bundles by sending the basis elements $e_{12}(0, k)$, $k \in \mathbb{Z}/(\lambda_2 - \lambda_1)\mathbb{Z}$, defined by (2.4) to the functions

$$\theta_{(\lambda_2-\lambda_1)\mathbb{Z},k}(z + y_{12}\tau - t_{12}, \frac{\tau}{\lambda_2 - \lambda_1})$$

regarded as holomorphic sections of the line bundle

$$(t^*_{y_1\tau - t_1}\mathcal{L} \otimes \mathcal{L}^{\otimes(\lambda_1-1)})^* \otimes (t^*_{y_2\tau - t_2}\mathcal{L} \otimes \mathcal{L}^{\otimes(\lambda_2-1)}) \simeq t^*_{y_{12}\tau - t_{12}}\mathcal{L}^{\otimes(\lambda_2-\lambda_1)}.$$

The fact that this map respects m_2 follows from addition formulas for theta functions. To extend this equivalence to all lines and all stable bundles we use isogenies. For every positive integer r consider the natural isogeny of degree r

$$\pi_r : E_{r\tau} \to E_\tau.$$

Then we have the natural functors π_{r*} and π_r^* between the categories of bundles on E_τ and $E_{r\tau}$. We complete the construction of our equivalence by requiring that these functors correspond to the obvious functors π_{r*} and π_r^* between the corresponding Fukaya categories (see [17] for details). One also has to identify morphisms of degree 1 in both categories. For this one has to fix a non-zero holomorphic 1-form on E_τ and use the isomorphisms $\mathrm{Hom}(V_1, V_2)^* \simeq \mathrm{Ext}^1(V_2, V_1)$ (where V_1 and V_2 are vector bundles on E_τ) induced by Serre duality together with the obvious isomorphisms $\mathrm{Hom}^0(L_1, L_2)^* \simeq \mathrm{Hom}^1(L_2, L_1)$ in the Fukaya category.

2.3. **Triple products and indefinite theta series.** Consider 4 lines ($L_i = L(\lambda_i, y_i), i \in \mathbb{Z}/4\mathbb{Z}$) where $\lambda_i \in \mathbb{Q}$, $y_i \in \mathbb{R}$. As before, we assume that the corresponding circles in $\mathbb{R}^2/\mathbb{Z}^2$ form a transversal configuration, in particular, the lines L_i are distinct modulo $\mathbb{Z}^2$ and $\lambda_i \neq \lambda_{i+1}$ for $i \in \mathbb{Z}/4\mathbb{Z}$. Let $t_i, i \in \mathbb{Z}/4\mathbb{Z}$ be some real numbers, then (L_i, t_i) are objects of the Fukaya category. We are going to compute the Fukaya triple product $m_3([P_{01}], [P_{12}], [P_{23}])$, where $P_{i,i+1} := P_{i,i+1}(y_i, y_{i+1})$ for $i = 0, 1, 2$. This product is zero unless the following equality is satisfied:

$$\sum_{i=0}^{2} \deg \mathrm{Hom}(L_i, L_{i+1}) = \deg \mathrm{Hom}(L_0, L_3) + 1. \tag{2.6}$$

Following the definition we have to consider all quadrangles (up to translation by $\mathbb{Z}^2$) with vertices p_i such that for every i the vector $p_i - p_{i-1}$ has slope λ_i, $p_i \equiv P_{i,i+1} \mod \mathbb{Z}^2$, for $i = 0, 1, 2$, and the piecewise linear path $[p_0, p_1, p_2, p_3]$ goes in the clockwise direction. First of all, it is easy to check that the condition

(2.6) implies that all such quadrangles are convex. Secondly, the condition on the orientation of the path is equivalent to the system of inequalities

$$\det(p_{i+1} - p_i, p_i - p_{i-1}) > 0 \tag{2.7}$$

where $i \in \mathbb{Z}/4\mathbb{Z}$. These quadrangles (considered up to translations by $\mathbb{Z}^2$) can be parametrized by elements of a rank-2 lattice. Namely, let

$$\Lambda = \Lambda(\lambda_0, \dots, \lambda_3) = \{\mathbf{n} = (n_0, \dots, n_3) \in \mathbb{Q}^4 : \sum n_i = \sum \lambda_i n_i = 0, n_1 \in I_{\lambda_1}, n_2 \in I_{\lambda_2}\}.$$

Then writing

$$p_i - p_{i-1} = x_i(1, \lambda_i)$$

for $i \in \mathbb{Z}/4\mathbb{Z}$ we obtain the vector $\mathbf{x} = (x_0, \dots, x_3)$ in $\Lambda_{\mathbb{R}}$. The inequalities (2.7) become

$$(\lambda_i - \lambda_{i+1})x_i x_{i+1} > 0 \tag{2.8}$$

for $i \in \mathbb{Z}/4\mathbb{Z}$. On the other hand, setting

$$P_{i,i+1} - P_{i-1,i} = v_i(1, \lambda_i)$$

we obtain the vector $\mathbf{v} = (v_0, \dots, v_i) \in \Lambda_{\mathbb{R}}$ (note that $v_i = y_{i,i+1} - y_{i-1,i}$). Now the conditions $p_i \equiv P_{i,i+1} \mod \mathbb{Z}^2$ for $i = 0, 1, 2$ imply that $\mathbf{x} - \mathbf{v}$ belongs to Λ. Conversely, given an element $\mathbf{n} = (n_0, \dots, n_3) \in \Lambda$ we have the corresponding quadrangle $\Delta(\mathbf{n})$ with vertices p_i such that $p_0 = P_{0,1}$ and $p_i - p_{i-1} = (v_i + n_i)(1, \lambda_i)$. Fixing the fourth vertex p_3 modulo $\mathbb{Z}^2$ is equivalent to choosing $\mathbf{n}$ in a fixed coset modulo the sublattice $\Lambda^+ \subset \Lambda$ defined as follows:

$$\Lambda^+ = \Lambda^+(\lambda_0, \dots, \lambda_3) = \{\mathbf{n} = (n_0, \dots, n_3) \in \mathbb{Z}^4 : \sum n_i = \sum \lambda_i n_i = 0, \lambda_i n_i \in \mathbb{Z}\}.$$

Thus, the sums in the definition of the Fukaya coefficients are taken over all elements $\mathbf{n}$ of a coset of Λ^+ in Λ, such that $\mathbf{v} + \mathbf{n} \in C$, where $C \subset \Lambda_{\mathbb{R}}$ is an open subset defined by inequalities (2.8). It is easy to see that the condition (2.6) implies that C is a non-empty open cone. The relation between the vertex p_3 and the coset $\mathbf{n} \in \Lambda/\Lambda^+$ can be found explicitly as follows. We know that $p_0 - p_3 = (v_0 + n_0)(1, \lambda_0)$, and that $p_0 \equiv P_{01}$. Hence, $p_3 \equiv P_{03}(y_0, y_3 + a\lambda_3 + b)$ where a and b are integers satisfying

$$\frac{a\lambda_3 + b}{\lambda_3 - \lambda_0} \equiv -n_0 \mod (I_{\lambda_0}).$$

It follows that

$$(a, b) \equiv (n_1 + n_2, -\lambda_1 n_1 - \lambda_2 n_2) \mod \Lambda(\lambda_0, \lambda_3).$$

The area of $\Delta(\mathbf{n})$ is given by

$$\int_{\Delta(\mathbf{n})} dx \wedge dy = \frac{1}{2}\left(\det(p_1 - p_0, p_0 - p_3) + \det(p_3 - p_2, p_2 - p_1)\right) = \frac{1}{2}Q(\mathbf{v} + \mathbf{n})$$

where Q is the quadratic form on $\Lambda_{\mathbb{R}}$ given by

$$Q(\mathbf{x}) = (\lambda_0 - \lambda_1)x_0 x_1 + (\lambda_2 - \lambda_3)x_2 x_3.$$

Finally, we have

$$\sum_{i \in \mathbb{Z}/4\mathbb{Z}} (x(p_i) - x(p_{i-1}))t_i = \sum_i t_i x_i = -\mathbf{w} \cdot \mathbf{x}$$

where $\mathbf{w} = (w_0, \dots, w_3) \in \Lambda_{\mathbb{R}}$, $w_i = t_{i,i+1} - t_{i-1,i}$, $t_{i,j} = \frac{t_j - t_i}{\lambda_j - \lambda_i}$, $\mathbf{x} \cdot \mathbf{x}'$ is the symmetric pairing induced by Q (so that $Q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x}$).

Thus, we have

$$m_3([P_{01}(y_0,y_1)],[P_{12}(y_1,y_2)],[P_{23}(y_2,y_3)]) =$$
$$\sum_{\mathbf{n}\in\Lambda,\mathbf{v}+\mathbf{n}\in C} \pm\exp(\pi i\tau Q(\mathbf{v}+\mathbf{n})-2\pi i\mathbf{w}\cdot(\mathbf{v}+\mathbf{n}))[P_{03}(y_0,y_3+(n_1+n_2)\lambda_3-\lambda_1 n_1-\lambda_2 n_2)]$$

where the sign is equal to the sign of v_0+n_0. Let us choose C^+ to be the component of C where $x_0>0$. Then using the bases $e_{ij}(m,n)$ defined by (2.4) we can rewrite the above formula as follows:

$$m_3(e_{01}(0,0),e_{12}(0,0),e_{23}(0,0)) =$$
$$\sum_{\mathbf{n}\in\Lambda/\Lambda^+} \Theta_{\Lambda^+,Q,C;\mathbf{n}}(\tau\mathbf{v}-\mathbf{w},\tau)e_{03}(n_1+n_2,-\lambda_1 n_1-\lambda_2 n_2)$$

where $\Theta_{\Lambda^+,Q,C;\mathbf{n}}$ is the indefinite theta series with characteristic $\mathbf{n}$ defined by (1.1). Similarly we can compute products of all basis elements:

$$m_3(e_{01}(a,b),e_{12}(c,d),e_{23}(f,g)) =$$
$$\sum_{\mathbf{n}\in\Lambda/\Lambda^+} \Theta_{\Lambda^+,Q,C;\mathbf{u}+\mathbf{n}}(\tau\mathbf{v}-\mathbf{w},\tau)e_{03}(a+c+f+n_1+n_2,b+d+g-\lambda_1 n_1-\lambda_2 n_2)\tag{2.9}$$

where $\mathbf{u}=(u_0,u_1,u_2,u_3)\in\Lambda_{\mathbb{Q}}$ is the following vector:

$$\mathbf{u}=(\frac{a\lambda_1+b}{\lambda_1-\lambda_0}-\frac{(a+c+f)\lambda_3+b+d+g}{\lambda_3-\lambda_0},\frac{c\lambda_2+d}{\lambda_2-\lambda_1}-\frac{a\lambda_0+b}{\lambda_1-\lambda_0},\frac{f\lambda_3+g}{\lambda_3-\lambda_2}-\frac{c\lambda_1+d}{\lambda_2-\lambda_1},$$
$$\frac{(a+c)\lambda_0+f\lambda_3+b+d+g}{\lambda_3-\lambda_0}-\frac{f\lambda_3+g}{\lambda_3-\lambda_2}).$$

The A_∞-axiom (2.1) can be converted into a certain identity for indefinite theta series (and the usual theta functions which appear from m_2). The explicit formula can be found in the last section of [13].

It is convenient for explicit computations to choose a pair of components (x_i,x_j) as coordinates on $\Lambda_{\mathbb{R}}$ in such a way that the cone C defined by inequalities (2.8) coincides with the cone $x_i x_j>0$. We will do this in two particular cases.

1. First assume that $\lambda_1<\lambda_0\le\lambda_2<\lambda_3$. Choose (x_0,x_1) as coordinates in $\Lambda_{\mathbb{R}}$. Then we have $C=\{\mathbf{x}:\ x_0x_1>0\}$. The quadratic form in these coordinates can be written as

$$Q(\mathbf{x})=ax_0^2+2bx_0x_1+cx_1^2,$$

where

$$a=\frac{(\lambda_2-\lambda_0)(\lambda_3-\lambda_0)}{\lambda_3-\lambda_2},$$
$$b=\frac{(\lambda_2-\lambda_1)(\lambda_3-\lambda_0)}{\lambda_3-\lambda_2},$$
$$c=\frac{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}{\lambda_3-\lambda_2}.$$

If $\lambda_0=\lambda_2$ (in this case we say that this is a *trapezoid* triple product) then $a=0$, so the form Q splits over $\mathbb{Q}$. On the other hand, if we set $\lambda_0=0$ then the corresponding transformation $(\lambda_1,\lambda_2,\lambda_3)\mapsto(a,b,c)$ is birational. More precisely, the inverse transformation is given by

$$\lambda_1=-\frac{D}{b},$$

$$\lambda_2 = \frac{aD}{b(b-a)},$$

$$\lambda_3 = \frac{D}{c-b}$$

where $D = b^2 - ac$. The inequalities satisfied by λ_i are equivalent to the following inequalities for a, b, c:

$$-D < 0 \le a < b < c.$$

2. Now assume that $\lambda_2 < \lambda_0 < \lambda_3 < \lambda_1$. Choose (x_0, x_3) as coordinates in $\Lambda_{\mathbb{R}}$. Then $C = \{\mathbf{x} : \ x_0 x_3 > 0\}$ and

$$Q(\mathbf{x}) = ax_0^2 + 2bx_0x_3 + cx_3^2,$$

where

$$a = \frac{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)}{\lambda_1 - \lambda_2},$$

$$b = \frac{(\lambda_1 - \lambda_0)(\lambda_3 - \lambda_2)}{\lambda_1 - \lambda_2},$$

$$c = \frac{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}{\lambda_1 - \lambda_2}.$$

Setting $\lambda_0 = 0$ we get a birational transformation coinciding with the previous one up to permutation of variables and signs. So the inverse map is given by $\lambda_1 = D/(b-c)$, $\lambda_2 = aD/b(a-b)$, $\lambda_3 = D/b$. The inequalities for λ_i are equivalent to the following inequalities:

$$-D < 0 < a, c < b.$$

It follows that every indefinite theta series associated with rational quadratic form of signature $(1,1)$ appears as a coefficient of certain Fukaya triple product. Indeed, let us consider the $\mathbb{Q}$-valued quadratic form $Q(x,y) = ax^2 + 2bxy + cy^2$ on $\mathbb{Z}^2$ such that $D = b^2 - ac > 0$ and Q is positive on the cone $C_0 = \{(x,y) \in \mathbb{R}^2 : \ xy > 0\}$. Assume first that $ac \neq 0$ and $b \neq c$. Then we necessarily have that a, b and c are positive and either $a < b$ or $c < b$. Permuting the coordinates if necessary we can assume that $a < b$. Then either $c > b$ and we are in the situation of the case 1 above or $c < b$ so we can apply the case 2. Note that the lattice coming from the configuration of lines will be commensurable with $\mathbb{Z}^2$, so we can use formulas of section 1 to relate the indefinite theta series associated with $(\mathbb{Z}^2, Q, C_0)$ to the corresponding Fukaya triple product. Furthermore, by rescaling the coordinates x, y we can always achieve that we are in the situation of a given case. For example, for the proof of Theorem 1 we will use a rescaling which leads to the case 1. If $a = 0$ and $b \neq c$ we can still apply the formulas of either case 1 or case 2. If $b = c$ then one of the slopes will be infinite. The only reason why we didn't include vertical lines in our category was because they correspond to torsion sheaves on elliptic curves while we want to deal only with bundles. However, the Fukaya compositions are well-defined for all lines including vertical, so a slight modification of the above computation will work in this case. Alternatively, we can always rescale the coordinates in such a way that $b \neq c$ and then apply the above formulas. On the other hand, we notice that replacing the form Q by NQ for some $N > 0$ we can always achive that all the slopes λ_i are integers.

Remark. For computations with the form Q on $\Lambda_{\mathbb{R}}$ defined above the following formula is useful:

$$\mathbf{x} \cdot \mathbf{y} = (\lambda_i - \lambda_{i+1}) x_i y_{i+1} + (\lambda_{i+2} - \lambda_{i+3}) y_{i+2} x_{i+3},$$

for any $i \in \mathbb{Z}/4\mathbb{Z}$, $\mathbf{x}, \mathbf{y} \in \Lambda_{\mathbb{R}}$.

3. Morphisms of vector bundles on elliptic curves

We identify an elliptic curve E with its dual by associating to a point $x \in E$ the line bundle $\mathcal{P}_x = \mathcal{O}_E(x - e)$ of degree zero on E, where $e \in E$ is the neutral element of the group law. For every integer d we denote by E_d the kernel of the homomorphism $[d] : E \to E : x \mapsto dx$.

Proposition 3.1. *Let V_1, V_2 and V_3 be stable vector bundles on an elliptic curve E. Assume that V_i has rank r_i and degree d_i and that the slopes $\mu_i = d_i/r_i$ satisfy $\mu_1 < \mu_2 < \mu_3$. Then there exists an integer d depending only on (d_i) and (r_i) such that the following natural map is surjective*

$$\oplus_{x \in E_d} \operatorname{Hom}(V_1, V_2 \otimes \mathcal{P}_x) \otimes \operatorname{Hom}(V_2 \otimes \mathcal{P}_x, V_3) \to \operatorname{Hom}(V_1, V_3).$$

Proof. Consider the action of $E \times E$ on the category of vector bundles on E, such that a point (x, y) acts as the functor $T_{(x,y)} : F \mapsto t_x^* F \otimes \mathcal{P}_y$. Then the statement of the theorem can be reformulated as follows: there exists a finite subgroup $S \subset E \times E$ such that the map

$$\oplus_{s \in S} \operatorname{Hom}(V_1, T_s(V_2)) \otimes \operatorname{Hom}(T_s(V_2), V_3) \to \operatorname{Hom}(V_1, V_3) \tag{3.1}$$

is surjective. Indeed, this follows from the fact that for any $x \in E$ one has

$$t_{r_2 x}^* V_2 \simeq V_2 \otimes \mathcal{P}_{-d_2 x}.$$

Now we claim that in proving the surjectivity of (3.1) we can replace the bundles V_i by $V_i \otimes L$, where L is a line bundle, or by $\mathcal{S}(V_i)$ provided that $d_1 > 0$, where $\mathcal{S}$ is the Fourier-Mukai transform (see [12]). Indeed, in the first case this is clear. In the case of the Fourier-Mukai transform this follows from the fact that $\mathcal{S}$ interchanges translations with tensoring by line bundles of degree zero.

Using these two operations (tensoring with a line bundle and the Fourier-Mukai transform) we can make $V_1 = \mathcal{O}_E$. Next we want to reduce the proof to the case when V_2 is a line bundle. Indeed, assume that in this case the assertion is true. Then consider an isogeny $\pi : E' \to E$ of degree r_2 and a line bundle L on E' such that $\pi_* L \simeq V_2$ (such π and L always exist). By assumption the statement is true for the triple $(\mathcal{O}_{E'}, L, \pi^* V_3)$ on E' (since $\pi^* V_3$ is a direct sum of stable bundles), hence there exists d such that the map

$$\oplus_{x \in E'_d} H^0(E', L \otimes \mathcal{P}_x) \otimes \operatorname{Hom}(L \otimes \mathcal{P}_x, \pi^* V_3) \to H^0(E', \pi^* V_3) \tag{3.2}$$

is surjective. Now we notice that for every $x \in E'$ there is a natural commutative diagram

$$\begin{array}{ccc} H^0(E', L\otimes\mathcal{P}_x)\otimes \mathrm{Hom}(L\otimes\mathcal{P}_x, \pi^*V_3) & \longrightarrow & H^0(E',\pi^*V_3) \\ \downarrow & & \downarrow \\ H^0(E,\pi_*(L\otimes\mathcal{P}_x))\otimes\mathrm{Hom}(\pi_*(L\otimes\mathcal{P}_x), V_3) & \longrightarrow & H^0(E,V_3) \end{array} \tag{3.3}$$

in which the right vertical arrow is the following composition of natural morphisms:

$$H^0(E',\pi^*V_3)\widetilde{\rightarrow} H^0(E,\pi_*\pi^*V_3)\rightarrow H^0(E,V_3).$$

Since V_3 is a direct summand in $\pi_*\pi^*V_3$, this map is surjective. Also for every $y \in E$ we have an isomorphism

$$\pi_*(L\otimes\mathcal{P}_{\hat{\pi}(x)})\simeq(\pi_*L)\otimes\mathcal{P}_x$$

where $\hat{\pi}: E \rightarrow E'$ is the isogeny dual to π. Hence, the surjectivity of (3.2) implies the surjectivity of the following map

$$\oplus_{y\in\hat{\pi}^{-1}(E'_d)}H^0(E,(\pi_*L)\otimes\mathcal{P}_y)\otimes\mathrm{Hom}((\pi_*L)\otimes\mathcal{P}_y, V_3)\rightarrow H^0(E,V_3)$$

as required.

It remains to prove the statement for a triple $(\mathcal{O}_E, L, V)$ where L is a line bundle, V is a stable bundle, such that $0 < \deg(L) < \mu(V)$. Note that $H^0(E,V)$ is an irreducible representation of the Heisenberg group H which is an extension of E_d by $\mathbb{G}_m$, where $d = \deg V$. More precisely, H is the group of pairs (x,ϕ) where $x \in E_d$, $\phi: V \rightarrow t_x^*V$. It follows that the image of the natural map

$$\oplus_{x\in E_d}H^0(E,t_x^*L)\otimes\mathrm{Hom}(t_x^*L,V)\rightarrow H^0(E,V)$$

is invariant under the H-action. Therefore, it suffices to prove that this map is not zero. Let $f: L \rightarrow V$ be a non-zero morphism. Then it is an injection of sheaves, hence the induced morphism $H^0(E,L)\rightarrow H^0(E,V)$ is injective which finishes the proof. □

Using the equivalence of the Fukaya category of $\mathbb{R}^2/\mathbb{Z}^2$ (without higher products) with the category of vector bundles on E we deduce the following corollary.

Corollary 3.2. *Let (L,t) (resp. (L',t')) be an object in the Fukaya category, where L (resp. L') is a line of slope λ (resp. λ'). Assume that $\lambda < \mu < \lambda'$, where $\mu \in \mathbb{Q}$. Then there exists a finite number of objects (M_i,t_i) in Fukaya category, where M_i are lines of slope μ, such that the composition*

$$m_2 : \oplus_i \mathrm{Hom}((L,t),(M_i,t_i))\otimes\mathrm{Hom}((M_i,t_i),(L',t'))\rightarrow\mathrm{Hom}((L,t),(L',t'))$$

is surjective. Furthermore, one can choose (M_i,t_i) in a generic position (i.e. in such a way that M_i do not pass through a finite number of given points).

4. Expression of all triple products via trapezoid ones

In this section we use the A_∞-axiom (2.1) to get a simple expression of an arbitrary triple product in $\mathcal{F}_s$ (corresponding to a transversal configuration of lines) in terms of trapezoid triple products and all double products.

First, consider the triple product $m_3(r,s,t)$ where $r \in \mathrm{Hom}^1(L_0,L_1)$, $s \in \mathrm{Hom}^0(L_1,L_2)$, $t \in \mathrm{Hom}^0(L_2,L_3)$, $\lambda_0 > \lambda_1 < \lambda_2 < \lambda_3$, $\lambda_0 < \lambda_3$. If $\lambda_2 = \lambda_0$ then this is a trapezoid product. Otherwise, there are two possibilities:
a) $\lambda_2 > \lambda_0$. In this case using Corollary 3.2 we can write s as a linear combination of products $s's''$ where $s' \in \mathrm{Hom}^0(L_1,L_0')$, $s'' \in \mathrm{Hom}^0(L_0',L_2)$, L_0' is a line of slope λ_0, $L_0' \neq L_0 \mod (\mathbb{Z}^2)$. Then applying the A_∞-constraint to the quadruple r,s',s'',t we obtain

$$m_3(r,s's'',t) = -m_3(r,s',s'')t + m_3(r,s',s''t)$$

(note that $rs' = 0$ by assumption while $m_3(s',s'',t) = 0$ as an element of $\mathrm{Hom}^{-1}(L_1,L_3)$).
b) $\lambda_2 < \lambda_0$. In this case we write t as a linear combination of products $t't''$ where $t' \in \mathrm{Hom}^0(L_2,L_0')$, $t'' \in \mathrm{Hom}^0(L_0',L_3)$, L_0' is a line of slope λ_0, $L_0' \neq L_0 \mod (\mathbb{Z}^2)$. Then we have $m_3(r,s,t') \in \mathrm{Hom}^0(L_0,L_0') = 0$. Hence, applying the A_∞-constraint to the quadruple r,s,t',t'' we get

$$m_3(r,s,t't'') = -m_3(rs,t',t'') + m_3(r,st',t'')$$

(note that $m_3(s,t',t'') = 0$).

One deals similarly with products of the type

$$\mathrm{Hom}^0(L_0,L_1) \otimes \mathrm{Hom}^0(L_1,L_2) \otimes \mathrm{Hom}^1(L_2,L_3) \to \mathrm{Hom}^0(L_0,L_3)$$

where $\lambda_0 < \lambda_1 < \lambda_2 > \lambda_3$.

Now let us consider $m_3(r,s,t)$ where $r \in \mathrm{Hom}^0(L_0,L_1)$, $s \in \mathrm{Hom}^1(L_1,L_2)$, $t \in \mathrm{Hom}^0(L_2,L_3)$, $\lambda_0 < \lambda_1 > \lambda_2 < \lambda_3$, $\lambda_0 < \lambda_3$. If $\lambda_1 = \lambda_3$ then this is a trapezoid product. Otherwise, there are two possibilities:
a) $\lambda_1 > \lambda_3$. Then we can write r as a linear combination of products $r'r''$, where $r' \in \mathrm{Hom}^0(L_0,L_3')$, $r'' \in \mathrm{Hom}^0(L_3',L_1)$, L_3' is a line of slope λ_3, $L_3' \neq L_3 \mod (\mathbb{Z}^2)$. Applying A_∞-constraint to (r',r'',s,t) we get

$$m_3(r'r'',s,t) = m_3(r',r''s,t) - m_3(r',r'',st) + m_3(r',r'',s)t$$

(since $m_3(r'',s,t) \in \mathrm{Hom}^0(L_3',L_3) = 0$). The first two terms in the RHS are trapezoid, while the product $m_3(r',r'',s)$ is of the form considered before.
a) $\lambda_1 < \lambda_3$. In this case we can write t as a linear combination of products $t't''$, where $t' \in \mathrm{Hom}^0(L_2,L_1')$, $t'' \in \mathrm{Hom}^0(L_1',L_3)$, L_1' is a line of slope λ_1, $L_1' \neq L_1 \mod (\mathbb{Z}^2)$. Applying A_∞-constraint to (r,s,t',t'') we get

$$m_3(r,s,t't'') = -m_3(rs,t',t'') + rm_3(s,t',t'') + m_3(r,s,t')t''$$

(since $st' \in \mathrm{Hom}^1(L_1,L_1') = 0$). The products $m_3(s,t',t'')$ and $m_3(r,s,t')$ are trapezoid, while the product $m_3(rs,t',t)$ is of the form considered before.

Finally, using the cyclic symmetry of m_3 we can reduce all non-zero transversal higher products m_3 to the ones considered above. For example, the product

$$\mathrm{Hom}^1(L_0,L_1) \otimes \mathrm{Hom}^1(L_1,L_2) \otimes \mathrm{Hom}^0(L_2,L_3) \to \mathrm{Hom}^1(L_0,L_3)$$

is equivalent to the product

$$\mathrm{Hom}^1(L_1,L_2) \otimes \mathrm{Hom}^0(L_2,L_3) \otimes \mathrm{Hom}^0(L_3,L_0) \to \mathrm{Hom}^0(L_1,L_0).$$

Proof of Theorem 1.
First, let us introduce some notation. We assume that for all objects (L,t) of the Fukaya category that appear below a representative of the line L modulo $\mathbb{Z}^2$-translations is fixed. Then for every pair of objects (L_1,t_1) and (L_2,t_2) and every intersection point $P \in (L_1+\mathbb{Z}^2) \cap (L_2+\mathbb{Z}^2)/\mathbb{Z}^2$ we represent P in the form

$P_{12}(y_1, y_2 + m\lambda_2 + n)$, where $L_i = L(\lambda_i, y_i)$, and set

$$e(P) = e_{12}(m, n)$$

where $e_{12}(m, n)$ is defined by formula (2.4).

As was explained in section 2.3 (after passing to a commensurable lattice Λ) we can assume that the lattice Λ, the quadratic form Q, and the cone C come from a quadruple of rational numbers $\lambda_0, \dots, \lambda_3$ such that $\lambda_1 < \lambda_0 < \lambda_2 < \lambda_3$. Let us represent the variable $\mathbf{z} \in \Lambda_{\mathbb{C}}$ in the form $\mathbf{z} = \tau\mathbf{v} - \mathbf{w}$ with $\mathbf{v}, \mathbf{w} \in \Lambda_{\mathbb{R}}$. Then the value of $\Theta_{\Lambda,Q,C}(\mathbf{z}, \tau)$ appears as the coefficient with $e(P_{03})$ of the triple product $m_3(e(P_{01}), e(P_{12}), e(P_{23}))$ for a quadruple of objects (L_i, t_i), $i = 0, \dots, 3$, and intersection points $P_{i,i+1} \in L_i \cap L_{i+1}$, where $L_0 = L(\lambda_0, 0)$, $L_1 = L(\lambda_1, 0)$, $L_2 = (\lambda_2, (\lambda_2 - \lambda_1)v_1)$, $L_3 = (\lambda_3, (\lambda_0 - \lambda_3)v_0)$, $t_0 = t_1 = 0$, $t_2 = (\lambda_2 - \lambda_1)w_1$, $t_3 = (\lambda_0 - \lambda_3)w_0$. Now we can apply the above procedure of expressing this triple product in terms of the trapezoid ones. More precisely, due to the inequalities $\lambda_1 < \lambda_0 < \lambda_2 < \lambda_3$ we apply the very first case of the above argument. This means that we choose a finite number of (not necessarily distinct) objects (M_j, t_j), where M_j are lines of slope λ_0 different from L_0, and intersection points $Q_j \in L_1 \cap (M_j + \mathbb{Z}^2)$, $R_j \in M_j \cap (L_2 + \mathbb{Z}^2)$ such that $m_2(e(Q_j), e(R_j))$ form a basis in $\mathrm{Hom}((L_1, t_1), (L_2, t_2))$. The transition matrix from this basis to the standard basis of intersection points of L_1 and L_2 modulo $\mathbb{Z}^2$ is given by elliptic functions of (z_1, τ). Thus, we can write

$$e(P_{12}) = \sum_j \phi_j m_2(e(Q_j), e(R_j))$$

where ϕ_j are meromorphic elliptic functions of (z_1, τ). Applying the A_∞-identity (2.1) we get

$$m_3(e(P_{01}), e(P_{12}), e(P_{23})) =$$
$$\sum_j \rho_j(-m_3(e(P_{01}), e(Q_j), e(R_j))e(P_{23}) + m_3(e(P_{0,1}), e(Q_j), e(R_j)e(P_{23}))).$$

Now by the results of section 2.3 and by formula (1.2) the coefficients of the trapezoid product $m_3(e(P_{0,1}), e(Q_j), e(R_j))$ are given (up to factors of the form $\exp(\pi i a\tau)$ with $a \in \mathbb{Q}_{>0}$) by the functions of the form $\kappa_e(a\tau + b, gz_1 + c\tau + d; h\tau)$, where $a, b, c, d, e, g \in \mathbb{Q}$, $h \in \mathbb{Q}_0$, Multiplying the result with $e(P_{2,3})$ means that we get some linear combination of the above functions with coefficients which are elliptic functions of $(s(\mathbf{z}), \tau)$ for some linear functional s. Finally the products $m_3(e(P_{0,1}), e(Q_j), e(R_j)e(P_{2,3}))$ are expressed via elliptic functions of $(s(\mathbf{z}), \tau)$ and the functions of the form $\kappa_e(a\tau + b, gz_2 + c\tau + d; h\tau)$. $\square$

Remarks. 1. Since after rescaling Q the slopes λ_i can always be chosen to be integers one can replace the reference to Proposition 3.1 in the above proof by the well-known surjectivity statement for morphisms between line bundles.

2. It may seem strange that in Theorem 1 we substitute only constants in the second argument of κ. However, the function κ satisfies some identities (see [7], p. 481, formula (45) and the next one, or [13], formula (3.4.3)) which imply that one can express $\kappa(y, x; \tau)$ in terms of τ-elliptic functions and the function $\kappa(c, x + y - c; \tau)$ for any $c \in \mathbb{Q} + \mathbb{Q}\tau$.

5. Example

In this section we will give an example of identity produced by Theorem 1. Let us fix $a \in \mathbb{Z}$ such that $a \geq 2$ and consider the quadratic form Q on $\mathbb{Z}^2$ given by

$$Q(n_0, n_1) = an_0^2 + 4an_0n_1 + (4a-2)n_1^2.$$

Let us also consider the following split quadratic forms:

$$Q^1(n_0, n_1) = 2(n_0 + n_1)n_1$$

on $\mathbb{Z}^2$ and

$$Q^2(n_0, n_1) = (2n_0 + \frac{2a-1}{a}n_1)n_1$$

on the lattice $\Lambda^2 = \{(n_0, n_1) : \ n_0 \in \mathbb{Z}, n_1 \in a\mathbb{Z}\}$. As a cone C in all three cases we choose $x_0x_1 > 0$ (with $x_0 > 0$ in C^+) and set

$$\Theta(z_0, z_1) = \Theta_{\mathbb{Z}^2, Q, C}(z_0, z_1; \tau),$$

$$\Theta^1(z_0, z_1) = \Theta_{\mathbb{Z}^2, Q^1, C}(z_0, z_1; \tau),$$

$$\Theta^2_{c_0,c_1}(z_0, z_1) = \Theta_{\Lambda^2, Q^2, C; (c_0, c_1)}(z_0, z_1; \tau)$$

(we omit the variable τ in notation for brevity). Let us also denote by $\Delta(z)$ the determinant of the 2×2 matrix

$$(\theta_{2\mathbb{Z},i}(z + \frac{j}{2}, \frac{\tau}{2}))_{i \in \mathbb{Z}/2\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z}}.$$

Then we have the following identity:

$$\Delta(z_1 - \frac{\tau}{2})\Theta(z_0, z_1) = \theta_{2\mathbb{Z},1}(z_1 - \frac{\tau-1}{2}, \frac{\tau}{2}) \times$$
$$\{-\theta_{a\mathbb{Z}}(z_0 + 2z_1, \frac{\tau}{a})\Theta^1(-2z_1, \frac{\tau}{2}) + \sum_{l \in \mathbb{Z}/a\mathbb{Z}} \theta_{a\mathbb{Z},l}(-z_0 - 2z_1 + \frac{\tau}{2a}, \frac{\tau}{a})\Theta^2_{\frac{a-1}{a}l,-l}(z_0, \frac{\tau}{2})\}$$
$$- \theta_{2\mathbb{Z},1}(z_1 - \frac{\tau}{2}, \frac{\tau}{2}) \times$$
$$\{-\theta_{a\mathbb{Z}}(z_0 + 2z_1, \frac{\tau}{a})\Theta^1(-2z_1, \frac{\tau-1}{2}) + \sum_{l \in \mathbb{Z}/a\mathbb{Z}} \theta_{a\mathbb{Z},l}(-z_0 - 2z_1 + \frac{\tau-1}{2a}, \frac{\tau}{a})\Theta^2_{\frac{a-1}{a}l,-l}(z_0, \frac{\tau-1}{2})\}.$$

Let us write the variables in the form $z_i = v_i\tau - w_i$, where $i = 0, 1$, and consider the following objects in the Fukaya category: $(L_0 = L(0,0), t_0 = 0)$, $(L_1 = L(-1, 0), t_1 = 0)$, $(L_2 = L(1, 2v_1), t_2 = 2w_1)$, $(L_3 = L(\frac{a}{a-1}, -\frac{a}{a-1}v_0), t_3 = -\frac{a}{a-1}w_0)$. Then the series $\Theta(z_0, z_1)$ is equal to the coefficient with e_{03} in the triple product $m_3(e_{01}, e_{12}, e_{23})$ where we set $e_{ij} = e_{ij}(0, 0)$. Now we consider two auxiliary objects in the Fukaya category: $(M_0 = L(0, \frac{1}{2}), 0)$ and $(M_1 = L(0, \frac{1}{2}), \frac{1}{2}dx)$. There are unique points of intersection $Q_i \in L_1 \cap M_i$, $R_i \in M_i \cap L_2$, $i = 0, 1$. Note that the points Q_0 and Q_1 (resp. R_0 and R_1) coincide but we denote them differently since they belong to morphism spaces between different objects in the Fukaya category. The first step is to represent e_{12} as a linear combination of $e(Q_0)e(R_0)$ and $e(Q_1)e(R_1)$. We have

$$e(Q_i)e(R_i) = \theta_{2\mathbb{Z}}(z_1 - \frac{\tau - i}{2}, \frac{\tau}{2})e_{12} + \theta_{2\mathbb{Z},1}(z_1 - \frac{\tau}{2}, \frac{\tau - i}{2})e_{12}(0, 1)$$

for $i = 0, 1$. Hence,

$$\Delta(z_1 - \frac{\tau}{2})e_{12} = \theta_{2\mathbb{Z},1}(z_1 - \frac{\tau-1}{2}, \frac{\tau}{2})e(Q_0)e(R_0) - \theta_{2\mathbb{Z},1}(z_1 - \frac{\tau}{2}, \frac{\tau}{2})e(Q_1)e(R_1).$$

Next we use the formula

$$m_3(e_{01}, e(Q_i)e(R_i), e_{23}) = -m_3(e_{01}, e(Q_i), e(R_i))e_{23} + m_3(e_{01}, e(Q_i), e(R_i)e_{23}).$$

We have

$$m_3(e_{01}, e(Q_i), e(R_i)) = \Theta^1(-2z_1, \frac{\tau - i}{2})e_{02},$$
$$e(R_i)e_{23} = \sum_{l \in \mathbb{Z}/a\mathbb{Z}} \theta_{a\mathbb{Z},l}(-z_0 - 2z_1 + \frac{\tau - i}{2a}, \frac{\tau}{a})e^i_{03}(0, -l)$$

where $i = 1, 2$, $e^i_{03}(0, l)$ are the basis elements in $\mathrm{Hom}(M_i, L_3)$ defined by (2.4). The coefficient with e_{03} in the product $e_{02}e_{03}$ is equal to

$$\theta_{a\mathbb{Z}}(z_0 + 2z_1, \frac{\tau}{a}).$$

One more computation shows that the coefficient with e_{03} in the product $m_3(e_{01}, e(Q_i), e^i_{03}(0, -l))$ (where $i = 1, 2$) is equal to

$$\Theta^2_{\frac{a-1}{a}l, -l}(z_0, \frac{\tau - i}{2}).$$

Combining all these calculations we get the identity above.

6. Massey products

In this section we consider a family of well-defined univalued triple Massey products which define global sections of certain line bundles on the second cartesian power of the universal curve over the moduli stack of elliptic curves with some level structure.

6.1. **Definition of triple Massey products.** Let V_i, $0 \le i \le 3$ be holomorphic vector bundles on a complex manifold. Let $\alpha_1 \in \mathrm{Hom}(V_0, V_1)$, $\alpha_2 \in \mathrm{Ext}^1(V_1, V_2)$, $\alpha_3 \in \mathrm{Hom}(V_2, V_3)$ be elements satisfying $\alpha_2 \circ \alpha_1 = 0$, $\alpha_3 \circ \alpha_2 = 0$. Below we recall two equivalent constructions of the triple Massey product $MP(\alpha_1, \alpha_2, \alpha_3)$ which belongs to the cokernel of the morphism

$$\mathrm{Hom}(V_0, V_2) \oplus \mathrm{Hom}(V_1, V_3) \to \mathrm{Hom}(V_0, V_3) : (\beta_1, \beta_2) \mapsto \alpha_3 \circ \beta_1 + \beta_2 \circ \alpha_1. \tag{6.1}$$

Let us represent α_2 by a $\overline{\partial}$-closed $(0, 1)$-form $\widetilde{\alpha}_2$ with values in $V_1^* \otimes V_2$. Then by our assumption we have

$$\widetilde{\alpha}_2 \circ \alpha_1 = \overline{\partial}(\alpha_{12}),$$
$$\alpha_3 \circ \widetilde{\alpha}_2 = \overline{\partial}(\alpha_{23})$$

for some sections $\alpha_{12} \in C^\infty(V_1^* \otimes V_2)$, $\alpha_{23} \in C^\infty(V_2^* \otimes V_3)$. Now we set

$$MP(\alpha_1, \alpha_2, \alpha_3) = \alpha_3 \circ \alpha_{12} - \alpha_{23} \circ \alpha_1.$$

The ambiguity in a choice of α_{12} and α_{23} precisely means that $MP(\alpha_1, \alpha_2, \alpha_3)$ is correctly defined modulo the image of the map (6.1).

In the second definition [2] we consider an extension

$$0 \to V_2 \xrightarrow{i} V \xrightarrow{p} V_1 \to 0$$

with the class α_2. By our assumption there exist morphisms $\alpha_1' : V_0 \to V$ and $\alpha_3' : V \to V_3$ such that

$$\alpha_1 = p \circ \alpha_1',$$

[2] This definition is a particular case of the general construction of Massey products in triangulated categories, cf. [5],IV.2

$$\alpha_3 = \alpha_3' \circ i.$$

Now the composition $\alpha_3' \circ \alpha_1' \in \mathrm{Hom}(V_0, V_3)$ is well-defined modulo the image of (6.1).

Proposition 6.1. *One has*

$$MP(\alpha_1, \alpha_2, \alpha_3) = -\alpha_3' \circ \alpha_1'$$

in the cokernel of the map (6.1).

Proof. A choice of a closed $(0,1)$-form $\widetilde{\alpha}_2$ representing α_2 leads to the choice of V as follows. We set $V = V_2 \oplus V_1$ as a C^∞-bundle and define a holomorphic structure on it by the following $\overline{\partial}$-operator:

$$\overline{\partial}_V = \begin{pmatrix} \overline{\partial}_{V_2} & \widetilde{\alpha}_2 \\ 0 & \overline{\partial}_{V_1} \end{pmatrix}.$$

Let $\sigma : V_1 \to V$ be the natural C^∞-splitting arising from this description (so $p \circ \sigma = \mathrm{id}_{V_1}$). Then $\overline{\partial}(\sigma) = i \circ \widetilde{\alpha}_2$. Define $\rho : V \to V_2$ by the condition

$$i \circ \rho = \sigma \circ p - \mathrm{id}_V .$$

Then we have

$$\overline{\partial}(\rho) = \widetilde{\alpha}_2 \circ p.$$

Thus, we can choose

$$\alpha_{12} = \rho \circ \alpha_1',$$
$$\alpha_{23} = \alpha_3' \circ \sigma.$$

Hence,

$$\alpha_3 \circ \alpha_{12} - \alpha_{23} \circ \alpha_1 = \alpha_3' \circ i \circ \rho \circ \alpha_1' - \alpha_3' \circ \sigma \circ p \circ \alpha_1' = -\alpha_3' \circ \alpha_1'.$$

□

We will be interested in a particular case when $\mathrm{Hom}(V_0, V_2) = \mathrm{Hom}(V_1, V_3) = 0$. In this case the Massey product $MP(\alpha_1, \alpha_2, \alpha_3)$ is an element of $\mathrm{Hom}(V_0, V_3)$. Homological mirror conjecture for elliptic curve $E = E_\tau$ (proven in [16] for transversal products) implies that $MP(\alpha_1, \alpha_2, \alpha_3)$ is equal to the corresponding triple Fukaya product $m_3(\alpha_1, \alpha_2, \alpha_3)$. Indeed, the corresponding products are homotopic but in our case the homotopy takes values in zero spaces (see [15], sec.1.1, for a more detailed explanation).

Remark. The notion of transversality considered in [16] has to be strengthened, since the definition of Fukaya products m_k given in section 2.1 requires that no three of the corresponding circles intersect in one point (this was overlooked in [16]). However, all the proofs of [16] can be easily modified accordingly.

6.2. **Massey products for line bundles.** Let us fix a quadruple of integers (d_0, d_1, d_2, d) such that $1 \leq d_0 < \min(d_1, d_2)$, $d_0 + d = d_1 + d_2$. Let $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ be line bundles on elliptic curve E of degrees d_0, d_1, d_2 and d respectively, such that $\mathcal{L}_0\mathcal{L} \simeq \mathcal{L}_1\mathcal{L}_2$ (here and below we skip the sign of tensor product between line bundles for brevity). Then for any pair of sections $s_1 \in H^0(E, \mathcal{L}_1)$, $s_2 \in H^0(E, \mathcal{L}_2)$ and an element $e \in H^1(\mathcal{L}^{-1}) = H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1}\mathcal{L}_2^{-1})$ such that the compositions $s_1 e \in H^1(E, \mathcal{L}_0\mathcal{L}_2^{-1})$ and $e s_2 \in H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1})$ are zero, the triple Massey product $MP(s_1, e, s_2)$ defined in 6.1 is an element of $H^0(E, \mathcal{L}_0)$ (since

$H^0(E, \mathcal{L}_0\mathcal{L}_1^{-1}) = H^0(E, \mathcal{L}_0\mathcal{L}_2^{-1}) = 0)$. Thus, if we denote by $K_{s_1,s_2} \subset H^1(E, \mathcal{L}^{-1})$ the kernel of the natural map

$$H^1(E, \mathcal{L}^{-1}) \to H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1}) \oplus H^1(E, \mathcal{L}_0\mathcal{L}_2^{-1}) : e \mapsto (s_1 e, e s_2)$$

then the Massey product defines a linear map

$$MP : K_{s_1,s_2} \to H^0(E, \mathcal{L}_0). \tag{6.2}$$

Assume that $s_1 \neq 0$, $s_2 \neq 0$. Let D be the divisor of common zeroes of s_1 and s_2. Let us denote $\mathcal{L}'_i = \mathcal{L}_i(-D)$ for $i = 0, 1, 2$, $\mathcal{L}' = \mathcal{L}(-D)$, so that we still have $\mathcal{L}'_0\mathcal{L}' \simeq \mathcal{L}'_1\mathcal{L}'_2$. Then we have the induced sections $s'_1 \in H^0(E, \mathcal{L}'_1)$ and $s'_2 \in H^0(E, \mathcal{L}'_2)$ and a canonical map

$$\varphi : K_{s_1,s_2} \to K_{s'_1,s'_2}$$

induced by the morphism $\mathcal{L}^{-1} \to \mathcal{L}^{-1}(D) = (\mathcal{L}')^{-1}$. On the other hand, we can consider $H^0(E, \mathcal{L}'_0) = H^0(E, \mathcal{L}_0(-D))$ as a subspace in $H^0(E, \mathcal{L}_0)$.

Lemma 6.2. *For any $e \in K_{s_1,s_2}$ one has*

$$MP(s_1, e, s_2) = MP(s'_1, \varphi(e), s'_2).$$

Proof. First let us notice that

$$H^0(E, \mathcal{L}'_0(\mathcal{L}'_i)^{-1}) = H^0(E, \mathcal{L}_0\mathcal{L}_i^{-1}) = 0$$

for $i = 1, 2$, so $MP(s'_1, \varphi(e), s'_2)$ is defined. Let

$$0 \to \mathcal{L}^{-1} \to E \to \mathcal{O}_E \to 0$$

be an extension representing e. Let $f : \mathcal{L}_1^{-1} \to E$ be the lifting of $s_1 : \mathcal{L}_1^{-1} \to \mathcal{O}_E$. Then the image of f belongs to the following subextension E':

$$0 \to \mathcal{L}^{-1} \to E' \to \mathcal{O}_E(-D) \to 0.$$

Note that this extension represents the class $\varphi(e)$. Let $g' : E' \to \mathcal{L}_0\mathcal{L}_1^{-1}(-D)$ be the lifting of the map $s'_2 : \mathcal{L}^{-1} \to \mathcal{L}_0\mathcal{L}_1^{-1}(-D)$. Then according to Proposition 6.1 we have

$$MP(s'_1, \varphi(e), s'_2) = -g' \circ f.$$

On the other hand, the push-out of the extension E by the morphism $\mathcal{L}^{-1} \to \mathcal{L}^{-1}(D)$ coincides with extension

$$0 \to \mathcal{L}^{-1}(D) \to E'(D) \to \mathcal{O}_E \to 0.$$

Thus, we have an embedding $i : E \to E'(D)$, such that $i|'_E : E' \to E'(D)$ is the natural map. In particular, we can take $g = g' \circ i : E \to \mathcal{L}_0\mathcal{L}_1^{-1}$ as the lifting of the map $s_2 : \mathcal{L}^{-1} \to \mathcal{L}_0\mathcal{L}_1^{-1}$. Applying Proposition 6.1 again we obtain

$$MP(s_1, e, s_2) = -g \circ f$$

which finishes the proof. □

Thus, it suffices to study the case when the sections s_1 and s_2 have no common zeroes. In this case one has an exact sequence

$$0 \to \mathcal{L}_1^{-1}\mathcal{L}_2^{-1} \xrightarrow{\alpha} \mathcal{L}_1^{-1} \oplus \mathcal{L}_2^{-1} \xrightarrow{\beta} \mathcal{O}_E \to 0 \tag{6.3}$$

where $\alpha(s) = (-s s_2, s s_1)$, $\beta(t_1, t_2) = t_1 s_1 + t_2 s_2$. Tensoring by $\mathcal{L}_0$ and considering the corresponding sequence of cohomologies we get the following exact complex C:

$$0 \to H^0(E, \mathcal{L}_0) \xrightarrow{\delta_{-1}} H^1(E, \mathcal{L}^{-1}) \xrightarrow{\delta_0} H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1}) \oplus H^1(E, \mathcal{L}_0\mathcal{L}_2^{-1}) \to 0$$

Proposition 6.3. *For any $t \in H^0(E, \mathcal{L}_0)$ one has*

$$MP(s_1, \delta_{-1}(t), s_2) = t.$$

Proof. The class $\delta_{-1}(t) \in \mathrm{Ext}^1(\mathcal{L}_0^{-1}, \mathcal{L}_1^{-1}\mathcal{L}_2^{-1})$ is represented by the extension

$$0 \to \mathcal{L}_1^{-1}\mathcal{L}_2^{-1} \to E \to \mathcal{L}_0^{-1} \to 0$$

where $E = \beta^{-1}(t(\mathcal{L}_0^{-1}))$. The morphism $\mathcal{L}_0^{-1}\mathcal{L}_1^{-1} \xrightarrow{s_1} \mathcal{L}_0^{-1}$ lifts to the morphism

$$f : \mathcal{L}_0^{-1}\mathcal{L}_1^{-1} \to E : s \mapsto (st, 0),$$

while the morphism $\mathcal{L}_1^{-1}\mathcal{L}_2^{-1} \xrightarrow{s_2} \mathcal{L}_1^{-1}$ lifts to the morphism

$$g : E \to \mathcal{L}_1^{-1} : (t_1, t_2) \mapsto -t_1.$$

According to Proposition 6.1 we have

$$MP(s_1, \delta_{-1}(t), s_2) = -g \circ f,$$

so the result follows from the above formulas for f and g. □

The spaces K_{s_1,s_2} can be considered as stalks of the sheaf $\mathcal{K}$ over the appropriate moduli stack. The Massey product gives a morphism from $\mathcal{K}$ to the bundle with the fibre $H^0(E, \mathcal{L}_0)$. The previous proposition shows that over an open part this is an isomorphism inverse to δ_{-1}. In the section 7.1 we will compute the map MP in terms of indefinite theta series using (2.9) and applying the above proposition we will get a proof of Theorem 4. In order to get modular (or Jacobi) forms from the coefficients of the map MP we can try to map various standard line bundles (trivialized by theta functions) to $\mathcal{K}$. Below we will present two ways to do it. The first uses the determinants and gives Jacobi forms. The second approach (see section 7.2) is more direct and produces modular forms but it requires additional assumptions on the integers (d_0, d_1, d_2).

6.3. **Determinantal approach.** Let us consider the 1-dimensional vector space

$$M = \det H^1(E, \mathcal{L}^{-1}) \otimes \det H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1})^* \otimes \det H^1(E, \mathcal{L}_0\mathcal{L}_2^{-1})^*$$

where for a vector space V we denote by $\det V$ its top-degree wedge power. We claim that for fixed s_1 and s_2 there is a canonical map

$$e_{s_1,s_2} : M \otimes \bigwedge^{d_0-1} H^1(E, \mathcal{L}^{-1})^* \to K_{s_1,s_2}$$

Indeed, in general for a linear map $f : V \to W$ between vector spaces of dimensions n and $n-k$ one can construct a linear map

$$k_f : \det V \otimes \det W^* \otimes \bigwedge^{k-1} V^* \to V$$

such that its image belongs to $\ker(\phi)$ as follows. Start with the morphism

$$\bigwedge^{n-k} f^* : \det W^* = \bigwedge^{n-k} W^* \to \bigwedge^{n-k} V^*$$

and then consider the following composition

$$\phi : \det W^* \otimes \bigwedge^{k-1} V^* \to \bigwedge^{n-k} V^* \otimes \bigwedge^{k-1} V^* \to \bigwedge^{n-1} V^*$$

where the second arrow is given by the wedge product. It remains to use the isomorphism $\bigwedge^{n-1} V^* \simeq V \otimes \det V^*$. It is easy to see that the image of k_f belongs

to $\ker(f)$. If f is not surjective then $k_f = 0$, otherwise, k_f surjects onto $\ker(f)$. The map e_{s_1,s_2} is defined by applying this construction to the map

$$\delta_0 : H^1(E, \mathcal{L}^{-1}) \to H^1(E, \mathcal{L}_0\mathcal{L}_2^{-1}) \oplus H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1})$$

induced by compositions with s_1 and s_2 (note that dimensions of the spaces are $d_1 + d + 2 - d_0$ and $d_1 + d_2 - 2d_0$). Composing e_{s_1,s_2} with the map (6.2) above we obtain a linear map

$$MP^{\det}_{s_1,s_2} : M \otimes \bigwedge^{d_0-1} H^1(E, \mathcal{L}^{-1})^* \to H^0(E, \mathcal{L}_0) : \xi \mapsto MP(s_1, e_{s_1,s_2}(\xi), s_2).$$

It is easy to see that if the sections s_1 and s_2 have a common zero then δ_0 is not surjective, hence, $MP^{\det}_{s_1,s_2} = 0$.

We refer to [4], Appendix A, for the definition of determinants of complexes.

Proposition 6.4. *Assume that the divisors of s_1 and s_2 do not intersect. Then the map $\pm MP^{\det}_{s_1,s_2}$ is equal to the composition*

$$M \otimes \bigwedge^{d_0-1} H^1(E, \mathcal{L}^{-1})^* \to M \otimes \bigwedge^{d_0-1} H^0(E, \mathcal{L}_0)^* \to H^0(E, \mathcal{L}_0)$$

where the first arrow is induced by δ_{-1}^* and the second arrow is induced by the isomorphism $M \simeq \det H^0(E, \mathcal{L}_0)$ given by the canonical trivialization of the determinant of the exact complex C.

Proof. This follows from Proposition 6.3 and from the following observation. For an exact complex of the form

$$0 \to K \xrightarrow{\delta_{-1}} V \xrightarrow{\delta_0} W \to 0$$

the map

$$k_{\delta_0} : \det V \otimes \det W^* \otimes \bigwedge^{\dim K - 1} V^* \to K$$

constructed above coincides (up to a sign) with the composition

$$\det V \otimes \det W^* \otimes \bigwedge^{\dim K-1} V^* \xrightarrow{\delta_0^*} \det V \otimes \det W^* \otimes \bigwedge^{\dim K-1} K^* \to K$$

where $\det V \otimes \det W^* \simeq \det K$ by the canonical trivialization of the determinant of this complex. $\square$

Corollary 6.5. *Assume that $d_0 = 1$. Fix some bases in the spaces $H^1(E, \mathcal{L}^{-1})$, $H^1(E, \mathcal{L}_0\mathcal{L}_i^{-1})$ and $H^0(E, \mathcal{L}_0)$. Then we have*

$$MP^{\det}_{s_1,s_2} = \pm \det(C, B)^{-1}$$

where B is the corresponding basis of the complex C.

6.4. **Serre duality in homological mirror symmetry.** The identification of Ext^1-spaces in the Fukaya category and the category of bundles on elliptic curve $E = E_\tau$ uses the Serre duality in the following form (see [16]):

$$\mathrm{Hom}(V_1, V_2) \otimes \mathrm{Ext}^1(V_2, V_1) \to \mathbb{C} : f \otimes (g d\overline{z}) \mapsto \int_E dz \wedge \mathrm{Tr}(f \circ g d\overline{z}).$$

In other words, this is a composition of the natural map

$$\mathrm{Hom}(V_1, V_2) \otimes \mathrm{Ext}^1(V_2, V_1) \to H^1(E, \mathcal{O}_E)$$

and the map

$$\phi : H^1(E, \mathcal{O}_E) \to \mathbb{C} : \alpha \mapsto \int_E dz \wedge \alpha$$

where α is a $(0,1)$-form. The map ϕ is in turn equal to the composition of the isomorphism

$$H^1(E, \mathcal{O}_E) \to H^1(E, \omega_E)$$

induced by the holomorphic 1-form $\omega_0 = 2\pi i dz$ and the functional

$$I : H^1(E, \omega_E) \to \mathbb{C} : \eta \mapsto -\frac{1}{2\pi i} \int_E \eta$$

where η is a $(1,1)$-form. The factor of $\frac{1}{2\pi i}$ in the definition of I is important because then I admits an algebraic definition (in particular, it is defined over the field of definition of E). In fact, we can define the functional $I : H^1(C, \omega_C) \to \mathbb{C}$ for any Riemann surface C by the same formula. Then the following property of I shows that it is algebraically defined.

Lemma 6.6. *For every point $p \in C$ let $e_p \in H^1(C, \omega_C)$ be the class defined by the boundary homomorphism $\mathbb{C} \simeq H^0(C, \mathcal{O}_p) \to H^1(C, \omega_C)$ coming from the exact sequence*

$$0 \to \omega_C \to \omega_C(p) \stackrel{\mathrm{Res}_p}{\to} \mathcal{O}_p \to 0.$$

Then $I(e_p) = 1$.

Proof. Let $U \subset C$ be an open disk containing p. Consider the Cech complex of ω_C associated with the covering $(U, C - p)$:

$$\mathcal{C}^\cdot : \omega_C(U) \oplus \omega_C(C - p) \to \omega_C(U - p)$$

where the differential sends (α_U, α_p) to $\alpha_U|_{U-p} - \alpha_p|_{U-p}$. Since $H^1(C - p, \omega_C) = H^1(U, \omega_C) = 0$ the complex $\mathcal{C}^\cdot$ computes the cohomology of ω_C. The residue map

$$\mathrm{Res}_p : \mathcal{C}^1 = \omega_C(U - p) \to \mathbb{C}$$

descends to the functional $I' : H^1(C, \omega_C) \to \mathbb{C}$. It suffices to prove that $I = I'$. To this end let us consider the Cech complex associated with the same covering and with the complex of sheaves $\Omega^{1,0} \stackrel{\overline{\partial}}{\to} \Omega^{1,1}$ on C:

$$\mathcal{CD}^\cdot : \Omega^{1,0}(U) \oplus \Omega^{1,0}(C-p) \stackrel{d_1}{\to} \Omega^{1,1}(U) \oplus \Omega^{1,1}(C-p) \oplus \Omega^{1,0}(U-p) \stackrel{d_2}{\to} \Omega^{1,1}(U-p)$$

where

$$d_1(\alpha_U, \alpha_p) = (\overline{\partial}\alpha_U, \overline{\partial}\alpha_p, \alpha_U|_{U-p} - \alpha_p|_{U-p}),$$
$$d_2(\eta_U, \eta_p, \beta) = \eta_U|_{U-p} - \eta_p|_{U-p} - \overline{\partial}\beta.$$

The complex $\mathcal{CD}^\cdot$ is concentrated in degrees $[0, 2]$. We have natural morphisms of complexes $\mathcal{C}^\cdot \to \mathcal{CD}^\cdot$ and $\Omega^{1,\cdot}(C) \to \mathcal{D}^\cdot$ inducing isomorphisms on cohomologies. Now we define the functional $\widetilde{I} : \mathcal{CD}^1 \to \mathbb{C}$ by the formula

$$\widetilde{I}(\eta_U, \eta_p, \beta) = -\frac{1}{2\pi i} \int_D \eta_U - \frac{1}{2\pi i} \int_{C-D} \eta_p + \frac{1}{2\pi i} \int_{\partial D} \beta$$

where $D \subset U$ is a smaller disk containing p. It is easy to check that $\widetilde{I} \circ d_1 = 0$, hence, $\widetilde{I}$ descends to a functional

$$\widetilde{I} : H^1(C, \omega_C) \simeq H^1(\mathcal{CD}^\cdot) \to \mathbb{C}.$$

If η is a global $(1,1)$-form then

$$\widetilde{I}(\eta|_U, \eta|_{C-p}, 0) = -\frac{1}{2\pi i}\int_C \eta,$$

hence $\widetilde{I} = I$. On the other hand, if α is a holomorphic 1-form on $U - p$ then

$$\widetilde{I}(0,0,\alpha) = \mathrm{Res}_p(\alpha),$$

hence $\widetilde{I} = I'$. □

6.5. **Boundary homomorphism.** Below we will need to calculate the matrices of δ_{-1} and δ_0 with respect to the standard bases of the terms of the complex C (coming from Serre duality and the bases of theta functions in the spaces of global sections) for any pair of sections s_1, s_2 with no common zeroes. The components of δ_0 are just the compositions with s_1 and s_2 so they are given by theta functions. On the other hand, the map

$$\delta_{-1} : H^0(E, \mathcal{L}_0) \to H^1(E, \mathcal{L}^{-1})$$

is the composition with the class $\delta(s_1, s_2) \in H^1(E, \mathcal{L}_1^{-1}\mathcal{L}_2^{-1})$ corresponding to the extension (6.3). Via Serre duality δ_{-1} corresponds to a bilinear form

$$B_{s_1,s_2} : H^0(E, \mathcal{L}_0) \otimes H^0(E, \mathcal{L}) \to H^0(E, \mathcal{L}_1\mathcal{L}_2) \to \mathbb{C}$$

induced by the functional on $H^0(E, \mathcal{L}_1\mathcal{L}_2)$ dual to the class $\delta(s_1, s_2)$ (recall that we always use the trivialization of ω_E given by the form $2\pi i dz$).

Lemma 6.7. *Assume that s_1 and s_2 have no common zeroes. Then one has*

$$B_{s_1,s_2}(s,t) = \sum_{x \in Z(s_1)} \mathrm{Res}_x\Big(\frac{2\pi i s(z)t(z)dz}{s_1(z)s_2(z)}\Big)$$

where $Z(s_1)$ is the divisor of zeroes of s_1.

Proof. Applying the octahedron axiom to the composition of arrows $\mathcal{L}_1^{-1} \to \mathcal{L}_1^{-1} \oplus \mathcal{L}_2^{-1} \xrightarrow{\beta} \mathcal{O}_E$ one can easily derive that the class $\delta(s_1, s_2)$ can be represented as the following composition:

$$\mathcal{O}_E \to \mathcal{O}_{Z(s_1)} \to \mathcal{L}_1^{-1}\mathcal{L}_2^{-1}[1]$$

where the first arrow is the canonical one, the second arrow comes from the exact triangle

$$\mathcal{L}_1^{-1}\mathcal{L}_2^{-1} \xrightarrow{s_1} \mathcal{L}_2^{-1} \to \mathcal{O}_{Z(s_1)} \to \mathcal{L}_1^{-1}\mathcal{L}_2^{-1}[1]$$

(here we use the natural trivialization of $\mathcal{L}_2|_{Z(s_1)}$ induced by s_2). In other words, this element in $\mathrm{Ext}^1(\mathcal{O}_{Z(s_1)}, \mathcal{L}_1^{-1}\mathcal{L}_2^{-1})$ corresponds to the functional on $H^0(E, \mathcal{L}_1\mathcal{L}_2|_{Z(s_1)})$ which at the point $x \in Z(s_1)$ is equal to

$$\mathrm{Res}_x\Big(\frac{2\pi i dz}{s_1(z)s_2(z)}\Big)$$

□

7. Calculations

In this section we will use homological mirror symmetry to relate the Massey products considered in section 6.2 to indefinite theta series. We keep the notations of the previous section.

7.1. **Proof of Theorem 4.** Let us fix a pair of complex numbers $v_1\tau - w_1$ and $v_2\tau - w_2$ where $v_i, w_i \in \mathbb{R}$. Now consider the following 4 objects in the Fukaya category: $(L_0 = L(0,0), 0)$, $(L_1 = L(d_1, d_1v_1), d_1w_1)$, $(L_2 = L(d_0 - d_2, d_2v_2), d_2w_2)$, $(L_3 = L(d_0, 0), 0)$. Let $\mathcal{L}_\tau$ be the basic line bundle on $E = E_\tau$ such that $\theta(z) = \theta(z,\tau)$ is a section of $\mathcal{L}_\tau$. Under the equivalence with the category of bundles on E our 4 objects correspond to $\mathcal{O}_E$, $\mathcal{L}_1$, $\mathcal{L}_0\mathcal{L}_2^{-1}$ and $\mathcal{L}_0$ respectively, where $\mathcal{L}_1 = t^*_{v_1\tau-w_1}\mathcal{L}_\tau^{\otimes d_1}$, $\mathcal{L}_2 = t^*_{-v_2\tau+w_2}\mathcal{L}_\tau^{\otimes d_2}$, $\mathcal{L}_0 = \mathcal{L}_\tau^{\otimes d_0}$. Let $\Lambda^+ = \Lambda^+(0, d_1, d_0 - d_2, d_0)$ be the corresponding rank 2 lattice. Let us denote by $\mathbf{x} = (x_0, x_1, x_2, x_3)$ the element of $\Lambda^+ \otimes \mathbb{C}$ such that $x_0 = v_1\tau - w_1$ and $x_3 = v_2\tau - w_2$ (thus, $x_1 = -\frac{(d-d_1)x_0 + d_2x_3}{d}$, $x_2 = -\frac{(d-d_2)x_3 + d_1x_0}{d}$).

We start by fixing bases in all the relevant vector spaces. According to section 2.2 we have a natural isomorphism

$$\mathrm{Hom}(L_0, L_1) \simeq H^0(E, \mathcal{L}_1)$$

which identifies the basis $(e_{01}(0,a), a \in \mathbb{Z}/d_1\mathbb{Z})$ with $(\theta_{d_1\mathbb{Z},a}(z + x_0, \frac{\tau}{d_1}), a \in \mathbb{Z}/d_1\mathbb{Z})$. Similarly,

$$\mathrm{Hom}(L_2, L_3) \simeq H^0(E, \mathcal{L}_2)$$

such that $e_{23}(0,a)$, $a \in \mathbb{Z}/d_2\mathbb{Z}$, corresponds to $\theta_{d_2\mathbb{Z},a}(z - x_3, \frac{\tau}{d_2})$. On the other hand,

$$\mathrm{Hom}^1(L_1, L_2) \simeq H^1(E, \mathcal{L}^{-1}) \simeq H^0(E, \mathcal{L})^*$$

in such a way that the basis $(e_{12}(0,a), a \in \mathbb{Z}/d\mathbb{Z})$ is dual to the basis $(\theta_{d\mathbb{Z},-a}(z + x_0 + x_1, \frac{\tau}{d}), a \in \mathbb{Z}/d\mathbb{Z})$. Finally, the space $\mathrm{Hom}^1(L_0, L_2) \simeq H^1(E, \mathcal{L}_0\mathcal{L}_2^{-1})$ (resp. $\mathrm{Hom}^1(L_1, L_3) \simeq H^1(E, \mathcal{L}_0\mathcal{L}_1^{-1})$) has the basis $(e_{02}(0,a), a \in \mathbb{Z}/(d_2 - d_0)\mathbb{Z})$ (resp. $(e_{13}(0,a), a \in \mathbb{Z}/(d_1 - d_0)\mathbb{Z})$) and the space $\mathrm{Hom}(L_0, L_3) \simeq H^0(E, \mathcal{L}_0)$ has the basis $(e_{03}(0,a), a \in \mathbb{Z}/d_0\mathbb{Z})$ identified with $(\theta_{d_0\mathbb{Z},a}(z, \frac{\tau}{d_0}), a \in \mathbb{Z}/d_0\mathbb{Z})$.

We are going to compute explicitly the map (6.2) for $s_1 = \theta_{d_1\mathbb{Z}}(z + x_0, \frac{\tau}{d_1}) \in H^0(E, \mathcal{L}_1)$ and $s_2 = \theta_{d_2\mathbb{Z}}(z - x_3, \frac{\tau}{d_2}) \in H^0(E, \mathcal{L}_2)$ (the corresponding morphisms in the Fukaya category are $e_{01} = e_{01}(0,0)$ and $e_{23} = e_{23}(0,0)$).

First, using the formula (2.5) one can easily compute the matrix of δ_0. Namely, we have

$$m_2(e_{01}, e_{12}(0,k)) = \sum_{n \in \mathbb{Z}/I_1} \theta_{I_1, -\frac{k}{d} - n}(p_1x_1, p_1\tau) e_{02}(0, k + dn),$$
$$m_2(e_{12}(0,k), e_{23}) = \sum_{n \in \mathbb{Z}/I_2} \theta_{I_2, \frac{k}{d} + n}(p_2x_2, p_2\tau) e_{13}(0, k + dn),$$

where $I_i = \mathbb{Z} \cap \frac{d-d_i}{d_i}\mathbb{Z}$, $p_i = \frac{d_i d}{d - d_i}$, $i = 1, 2$. In these formulas we fix a representative of k in $\mathbb{Z}$. The change of a representative corresponds to a shift of the summation variable n. Here is a better way to write these formulas which doesn't require a choice of a representative of k:

$$m_2(e_{01}, e_{12}(0,k)) = \sum_{n \in \mathbb{Z}/dI_1,\ n \equiv k(d)} \theta_{I_1, -\frac{n}{d}}(p_1x_1, p_1\tau) e_{02}(0, n),$$
$$m_2(e_{12}(0,k), e_{23}) = \sum_{n \in \mathbb{Z}/dI_2,\ n \equiv k(d)} \theta_{I_2, \frac{n}{d}}(p_2x_2, p_2\tau) e_{13}(0, n), \tag{7.1}$$

Note that the coefficient of $m_2(e_{01}, e_{12}(0,k))$ (resp. $m_2(e_{12}(0,k), e_{23})$) with $e_{02}(0,i)$ (resp. $e_{13}(0,j)$) is non-zero only if $k \equiv i(d, d - d_1)$ (resp. $k \equiv j(d, d - d_2)$). Now it

is easy to deduce that an element $e=\sum_{k\in\mathbb{Z}/d\mathbb{Z}} c_k e_{12}(0,k)$ (where $c_k\in\mathbb{C}$) belongs to the subspace $K_{s_1,s_2}=\ker(\delta_0)$ if and only if

$$\sum_{n\in\mathbb{Z}/dI_1,\ n\equiv i(d-d_1)} c_n\theta_{I_1,-\frac{n}{d}}(p_1x_1,p_1\tau)=0,$$
$$\sum_{n\in\mathbb{Z}/dI_2,\ n\equiv j(d-d_2)} c_n\theta_{I_2,\frac{n}{d}}(p_2x_2,p_2\tau)=0 \tag{7.2}$$

for all $i\in\mathbb{Z}/(d-d_1)\mathbb{Z}$ and $j\in\mathbb{Z}/(d-d_2)\mathbb{Z}$.

Assume first that x_1 and x_2 are sufficiently generic, so that the corresponding circles in $\mathbb{R}^2/\mathbb{Z}^2$ form a transversal configuration. Then we can use (2.9) to compute the relevant triple Fukaya products. Note that the projection $(n_0,n_1,n_2,n_3)\mapsto(n_0,n_3)$ defines an isomorphism of the lattice Λ^+ with the lattice

$$\Lambda_{d_1,d_2,d}=\{(m,n)\in\mathbb{Z}^2:\ d_1m\equiv d_2n(d)\}.$$

On the other hand, the projection $(n_0,n_1,n_2,n_3)\mapsto(n_1,n_2)$ maps Λ^+ onto Λ_{d_1,d_2,d_0}. Thus, we have

$$m_3(e_{01},e_{12}(0,k),e_{23})=$$
$$\sum_{(n_1,n_2)\in\mathbb{Z}^2/\Lambda_{d_1,d_2,d_0}}\Theta_{\Lambda_{d_1,d_2,d},Q,mn>0;(-\frac{k+d_2n_2-d_1n_1}{d_0}-n_1,\frac{k+d_2n_2-d_1n_1}{d_0}-n_2)}(x_0,x_3;\tau)$$
$$e_{03}(0,k+d_2n_2-d_1n_1)$$

for $k\in\mathbb{Z}/d\mathbb{Z}$, where the form Q is given by

$$Q(m,n)=\frac{1}{d}(d_1(d-d_1)m^2+2d_1d_2mn+(d-d_2)d_2n^2)$$

Let $(e_{03}^*(0,l),0\le l<d_0)$ be the dual basis to $(e_{03}(0,l))$. Then for $e=\sum_k c_ke_{12}(0,k)$ we have

$$\widetilde{F}_l:=\langle e_{03}^*(0,l),MP(s_1,e,s_2)\rangle=\sum_k c_k\langle e_{03}^*(0,l),m_3(e_{01},e_{12}(0,k),e_{23})\rangle=\sum_k C_{l,k}c_k$$

where $C_{l,k}$ is zero unless $k\equiv l(d_0,d_1,d_2)$ in which case

$$C_{l,k}=\Theta_{\Lambda_{d_1,d_2,d},Q,mn>0;(-\frac{l}{d_0}-i-n_1,\frac{l}{d_0}+i-n_2)}(x_0,x_3;\tau)$$

where n_1, n_2, and i are some integers satisfying

$$k+d_2n_2-d_1n_1=l+d_0i.$$

Denoting $m_1=-i-n_1$, $m_2=i-n_2$ we can rewrite this formula as follows:

$$C_{l,k}=\Theta_{\Lambda_{d_1,d_2,d},Q,mn>0;(-\frac{l}{d_0}+m_1,\frac{l}{d_0}+m_2)}(x_0,x_3;\tau) \tag{7.3}$$

where $(m_1,m_2)\in\mathbb{Z}^2/\Lambda_{d_1,d_2,d}$ is defined by the congruence

$$d_2m_2-d_1m_1\equiv k-l(d).$$

Thus, we have

$$\widetilde{F}_l = \sum_{(m_1,m_2)\in\mathbb{Z}^2/\Lambda_{d_1,d_2,d}} c_{d_2m_2-d_1m_1+l}\Theta_{\Lambda_{d_1,d_2,d},Q,mn>0;(-\frac{l}{d_0}+m_1,\frac{l}{d_0}+m_2)}(x_0,x_3;\tau) =$$

$$\sum_{(m,n)\in\mathbb{Z}^2} \epsilon(m,n)c_{d_2n-d_1m+l}\exp(\pi i\tau Q(m-\frac{l}{d_0},n+\frac{l}{d_0})-2\pi i[d_1x_1(m-\frac{l}{d_0})+d_2x_2(n+\frac{l}{d_0})]) =$$

$$\sum_{(m,n)\in\mathbb{Z}^2} \epsilon(-m,-n)c_{d_1m-d_2n+l}\exp(\pi i\tau Q(m+\frac{l}{d_0},n-\frac{l}{d_0})+2\pi i[d_1x_1(m+\frac{l}{d_0})+d_2x_2(n-\frac{l}{d_0})])$$

where $\epsilon(m,n) = 0$ unless $(m-\frac{l}{d_0}+\alpha(x_0))(n+\frac{l}{d_0}+\alpha(x_3)) > 0$ in which case $\epsilon(m,n) = \operatorname{sign}(m-\frac{l}{d_0}+\alpha(x_0))$. Now an easy computation shows that the conditions (7.2) are equivalent to the system of equations

$$\sum_{m\in\mathbb{Z}} c_{d_1m-d_2n_0+l}\exp(\pi i\tau Q(m+\frac{l}{d_0},n_0-\frac{l}{d_0})+2\pi i d_1x_1(m+\frac{l}{d_0})) = 0,$$

$$\sum_{n\in\mathbb{Z}} c_{d_1m_0-d_2n+l}\exp(\pi i\tau Q(m_0+\frac{l}{d_0},n-\frac{l}{d_0})+2\pi i d_2x_2(n-\frac{l}{d_0})) = 0,$$

where $m_0,n_0\in\mathbb{Z}$, $0\le l<d_0$. Therefore, in the notation of Theorem 4 we obtain

$$\widetilde{F}_l = \sum_{(m,n)\in\mathbb{Z}^2} \epsilon(-m,-n)c_{d_1m-d_2n+l}a_{m,n,l},$$

and the above system is equivalent to the condition (0.4). This implies that one can replace the summation scheme defining $\widetilde{F}_l$ to the summation over $m,n\ge 0$ and $m,n<0$ (with signs "minus" and "plus", respectively). Hence, we obtain $\widetilde{F}_l = F_l$ (the latter series is defined in the formulation of Theorem 4), i.e.,

$$\langle e_{03}^*(0,l), MP(s_1,e,s_2)\rangle = F_l. \tag{7.4}$$

Now assume that the sections s_1 and s_2 have no common zeroes. Then we claim that formula (7.4) holds without any further genericity assumption on x_1 and x_2. Indeed, as we have seen in section 6.2, when the data $(\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2,s_1,s_2)$ vary in such a way that s_1 and s_2 have no common zeroes, the vector spaces K_{s_1,s_2} can be viewed as fibers of a vector bundle on the space of parameters. Furthermore, the map $MP : K_{s_1,s_2}\to H^0(E,\mathcal{L}_0)$ varies continuously with parameters. Since F_l is also a continuous function of x_1, x_2 and (c_k) varying in the vector bundle defined by (0.4), we derive that equation (7.4) holds whenever s_1 and s_2 have no common zeroes.

Finally, we are going to combine the result of section 6.5 with Proposition 6.3 to derive the system of equations for F_l. We have

$$\delta_{-1}(e_{03}(0,l)) = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} D_{k,l}e_{12}(0,k),$$

where

$$D_{k,l} = B_{s_1,s_2}(\theta_{d_0\mathbb{Z},l}(z,\frac{\tau}{d_0}),\theta_{d\mathbb{Z},-k}(z+x_0+x_1,\frac{\tau}{d})).$$

The divisor $Z(s_1)$ consists of the points $(-x_0+z(a), a\in\mathbb{Z}/d_1\mathbb{Z})$ where

$$z(a) = \frac{\tau}{2}+\frac{1}{2d_1}+\frac{a}{d_1}$$

Therefore, according to Lemma 6.7 one has

$$D_{k,l}=\sum_{a\in\mathbb{Z}/d_1\mathbb{Z}}\mathrm{Res}_{-x_0+z(a)}\left(\frac{2\pi i\theta_{d_0\mathbb{Z},l}(z,\frac{\tau}{d_0})\theta_{d\mathbb{Z},-k}(z+x_0+x_1,\frac{\tau}{d})dz}{\theta_{d_1\mathbb{Z}}(z+x_0,\frac{\tau}{d_1})\theta_{d_2\mathbb{Z}}(z-x_3,\frac{\tau}{d_2})}\right)=$$

$$\sum_{a\in\mathbb{Z}/d_1\mathbb{Z}}\frac{2\pi i\theta_{d_0\mathbb{Z},l}(-x_0+z(a),\frac{\tau}{d_0})\theta_{d\mathbb{Z},-k}(x_1+z(a),\frac{\tau}{d})}{\theta'_{d_1\mathbb{Z}}(z(a),\frac{\tau}{d_1})\theta_{d_2\mathbb{Z}}(x_1+x_2+z(a),\frac{\tau}{d_2})}=$$

$$\frac{2\pi i}{d_1\theta'_{\mathbb{Z}}(\frac{d_1\tau+1}{2},d_1\tau)}\sum_{a\in\mathbb{Z}/d_1\mathbb{Z}}\frac{\theta_{d_0\mathbb{Z},l}(\frac{(d_2-d)x_1+d_2x_2}{d_0}+z(a),\frac{\tau}{d_0})\theta_{d\mathbb{Z},-k}(x_1+z(a),\frac{\tau}{d})}{\theta_{d_2\mathbb{Z}}(x_1+x_2+z(a),\frac{\tau}{d_2})},$$

where we denote $\theta'_{\mathbb{Z}}(z,\tau)=\frac{\partial}{\partial z}\theta_{\mathbb{Z}}(z,\tau)$. It is easy to see that this formula is equivalent to the formula for $D_{k,l}$ in Theorem 4. Now Proposition 6.3 implies that

$$e=\delta_{-1}(\sum_l F_l e_{03}(0,l))=\sum_{k,l}D_{k,l}F_l e_{12}(0,k).$$

Equating the coefficients with $e_{12}(0,k)$ we get the system of linear equations on F_l. It remains to notice that disjointness of the divisors of s_1 and s_2 is equivalent to the condition (0.3) of Theorem 4.

7.2. **Examples of modular indefinite theta series.** Let us assume that $(d-d_i)|d$ for $i=1,2$. Choose an integer f such that $f|d$ and $(d-d_i)|f$ for $i=1,2$ and set

$$c_k=\begin{cases}1, & k\equiv 0(f),\\ 0, & k\not\equiv 0(f)\end{cases}$$

Then the conditions (7.2) boil down to the following pair of equations:

$$\theta_{\frac{f}{d}\mathbb{Z}}(p_1x_1,p_1\tau)=0,$$
$$\theta_{\frac{f}{d}\mathbb{Z}}(p_2x_2,p_2\tau)=0.$$

Hence, we can set

$$x_i=s_i\cdot\frac{f\tau}{2d}+r_i\frac{d-d_i}{2fd_i},i=1,2,\tag{7.5}$$

where s_i, r_i are odd integers, to satisfy these equations. Now Theorem 4 implies that the series

$$-F_0=\sum_{(m,n)\in\mathbb{Z}^2,d_1m\equiv d_2n(f)}^{indef}\exp(\pi i\tau Q(m,n)+2\pi i(d_1x_1m+d_2x_2n))$$

multiplied by the appropriate factor $\exp(\pi ic\tau)$ with $c\in\mathbb{Q}$, is modular. More precisely, we can apply Theorem 4 directly only if the condition (0.3) is satisfied, i.e. the corresponding section s_1 and s_2 have no common zeroes. Otherwise, we first use Lemma 6.2 to reduce to this case.

Let us denote by f_0 the least common multiple of $d-d_1$ and $d-d_2$. Notice that the congruence $d_1m-d_2n\equiv 0(f)$ implies that $(d-d_1)m$ and $(d-d_2)n$ belong to $f_0\mathbb{Z}$. So we can take $\frac{(d-d_1)m}{f_0}$ and $\frac{(d-d_2)n}{f_0}$ as the new summation variables. Then

we get

$$-F_0 = \sum_{(m,n)\in\mathbb{Z}^2, m\equiv n(\frac{f}{f_0})}^{indef} \exp(\pi i\tau \frac{f_0^2}{d}(\frac{d_1}{d-d_1}m^2 + \frac{2d_1d_2}{(d-d_1)(d-d_2)}mn + \frac{d_2}{d-d_2}n^2) +$$

$$\pi i\tau\frac{ff_0}{d}(\frac{d_1}{d-d_1}s_1m + \frac{d_2}{d-d_2}s_2n) + \pi i\frac{f_0}{f}(r_1m+r_2n))$$

To find the factor $\exp(\pi ic\tau)$ above recall that for $N>0$ the functions

$$\exp(\pi i\tau N\lambda^2)\theta_{N\mathbb{Z},i}(\lambda\tau+\mu, \frac{\tau}{N})$$

where $i\in\mathbb{Z}$, $\lambda,\mu\in\mathbb{Q}$, $\lambda>0$, and

$$\exp(\pi i\tau N/4)\theta'_{\mathbb{Z}}(\frac{N\tau+1}{2}, N\tau)$$

are modular. Hence,

$$\exp(\pi i\tau(d_0(\frac{f}{2dd_0}((d_2-d)s_1+d_2s_2)+\frac{1}{2})^2+d(\frac{f}{2d}s_1+\frac{1}{2})^2-d_2(\frac{f}{2d}(s_1+s_2)+\frac{1}{2})^2-\frac{d_1}{4}))D_{k,l}$$

are modular. Simplifying we conclude that

$$\exp(\pi i\tau\frac{f^2}{4d^2d_0}(d_1(d_2-d)s_1^2+2d_1d_2s_1s_2+d_2(d_1-d)s_2^2))F_0 \tag{7.6}$$

is modular.

Proof of Theorem 2. Let a,b,c,p be positive integers such that $a|b$, $c|b$, $p|(b/a+1)$, $p|(c/a+1)$, $D=b^2-ac>0$. Then we set $d_1=b(b+a)$, $d_2=b(b+c)$, $d=(b+a)(b+c)$, $d_0=D$, $f=(b+a)(b+c)p/h$, where h is the greatest common divisor of $b/a+1$ and $b/c+1$, so that we are in the situation considered above. We can rewrite the series (7.6) using the change of variables

$$\tau = \frac{ach^2}{b(b+a)(b+c)}\tau'.$$

Then the above argument implies the modularity of the series

$$q^{\frac{p^2ac(2bs_1s_2-as_1^2-cs_2^2)}{8D}} \cdot \sum_{(m,n)\in\mathbb{Z}^2, m\equiv n(p)}^{indef} \zeta_{2p}^{r_1m+r_2n} q^{bmn+a\frac{m^2+mps_1}{2}+c\frac{n^2+nps_2}{2}},$$

where ζ_{2p} is a primitive root of 1 of order $2p$ and r_1,r_2,s_1,s_2 are odd integers. Note that we have

$$\zeta_{2p}^{r_1m+r_2n} = \begin{cases} \zeta_p^{\frac{r_1+r_2}{2}m}, & \text{if } m\equiv n(2p), \\ -\zeta_p^{\frac{r_1+r_2}{2}m}, & \text{if } m\equiv n+p(2p). \end{cases}$$

Thus, the above series is equal to

$$q^{\frac{p^2ac(2bs_1s_2-as_1^2-cs_2^2)}{8D}} \cdot \sum_{(m,n)\in\mathbb{Z}^2, m\equiv n(p)}^{indef} (-1)^{\frac{n-m}{p}} \zeta_p^{\frac{r_1+r_2}{2}m} q^{bmn+a\frac{m^2+mps_1}{2}+c\frac{n^2+nps_2}{2}}.$$

Since $\frac{r_1+r_2}{2}$ can be an arbitrary integer we derive that for any residue $r\in\mathbb{Z}/p\mathbb{Z}$ the series

$$q^{\frac{p^2ac(2bs_1s_2-as_1^2-cs_2^2)}{8D}} \cdot \sum_{(m,n)\in\mathbb{Z}^2, m\equiv n\equiv r(p)}^{indef} (-1)^{\frac{n-m}{p}} q^{bmn+a\frac{m^2+mps_1}{2}+c\frac{n^2+nps_2}{2}}$$

is modular. $\square$

Proof of Theorem 6. Let us denote the series F_l considered above by $F_l(s_1, s_2)$ to show their dependence on a pair of odd integers s_1, s_2. Note that using the change of variables from the proof of Theorem 2 we can identify f_{s_1,s_2} with $-F_0(s_1, s_2)$ multiplied by some power of q. By definition $F_l(s_1, s_2) = 0$ unless $l \in (d - d_1)\mathbb{Z} + (d - d_2)\mathbb{Z}$. On the other hand, an easy computation shows that for all $l_1, l_2 \in \mathbb{Z}$ one has

$$F_0(s_1 + 2\frac{h}{p}l_1, s_2 + 2\frac{h}{p}l_2) = \zeta \cdot F_{(d-d_1)l_2-(d-d_2)l_1}(s_1, s_2)$$

where ζ is a root of unity. The assumption in Theorem 6 is precisely the condition (0.3) for the pair x_1, x_2 defined by (7.5). By Theorem 4 at least one series among F_l is non-zero which implies our statement. $\square$

The simplest examples of identities that can be derived from the above computations are obtained in the case $d_0 = 1$. Then $d = d_1 + d_2 - 1$ so the conditions $(d-d_1)|d$, $(d-d_2)|d$ are satisfied only in the following cases (assuming that $d_1 \le d_2$): (i) $d_1 = d_2 = 2$; (ii) $d_1 = 2$, $d_2 = 3$; and (iii) $d_1 = 3$, $d_2 = 4$. In case (i) we obtain formula (0.5) taking $f = 1$. In case (ii) we get identities (0.6), (0.7) and (0.8). More precisely, the case $f = 4$ leads to (0.6) while in the case $f = 2$ we get identities (0.7) and (0.8) corresponding to the cases $s_1 \equiv s_2(4)$ and $s_1 \equiv -s_2(4)$. In case (iii) above the assumption (0.3) of Theorem 4 is never satisfied, so we do not get any new identity.

One can also consider the degenerate case $d_0 = d_1 < d_2 = d$ in the above picture. The coefficients of the Massey products in this case are given by the series (0.9). Application of the above analysis implies that this is a meromorphic Jacobi form of weight 1 as claimed in Theorem 8. On the other hand, we obtain its expression in terms of theta functions. The case $a = 1$ is well known (see [18], Section 486, or [9] (5.26), or [14]). In the case $a = 2$ we get for any odd s

$$\sum_{n\in\mathbb{Z}} \frac{q^{\frac{n^2+ns}{2}}}{1 - q^{2n}u} = \varphi(q^2)^3 \frac{\sum_{n\in\mathbb{Z}} q^{2n^2+n(s-2)}u^n}{(\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2-n}u^n)(\sum_{n\in\mathbb{Z}} q^{2n^2+n(s-2)})}, \tag{7.7}$$

where $\varphi(q) = \prod_{n\ge1}(1 - q^n)$.

7.3. **String functions for $A_1^{(1)}$.** In the paper [9] the authors discovered a relation between the string functions of irreducible highest weight representations of affine Lie algebra $\mathfrak{g}$ of type $A_1^{(1)}$ and Hecke's modular forms for certain indefinite quadratic modules. Namely, for every dominant weight Λ and every weight λ they define the *string function* $c_\lambda^\Lambda(\tau)$ which describes a part of the character of the irreducible $\mathfrak{g}$-module with highest weight Λ (see [9]). One of the formulas they get for this function ([9], bottom of page 258) is

$$\eta(\tau)^3 c_\lambda^\Lambda(\tau) = \sum_{k,n\in\frac{1}{2}\mathbb{Z}, k\equiv n(\mathbb{Z}), k\ge|n| \text{or} -k>|n|} (-1)^{2k}\, \mathrm{sign}(k + \frac{1}{4}) q^{(m+2)(k+A)^2-m(n+B)^2},$$

where m is the level of Λ (the value of Λ on the central generator), A and B are rational numbers determined by Λ and λ, such that $2(m + 2)A \pm 2mB$ are odd integers. Taking $k - n$ and $k + n$ as new summation variables we can rewrite this

series as follows:

$$q^{\frac{2bs_1s_2-s_1^2-s_2^2}{8(b^2-1)}}\cdot\sum_{k,n\in\mathbb{Z}}^{indef}(-1)^{k+n}q^{(m+1)kn+\frac{k^2+s_1k}{2}+\frac{n^2+s_2n}{2}},$$

where $s_1=2(m+2)A+2mB$, $s_2=2(m+2)A-2mB$. As shown in [9] this series is equal to

$$\Theta^H_{\mathbb{Z}^2,Q;(A,B)},$$

where the quadratic form Q on $\mathbb{Z}^2$ is given by $Q(x,y)=2(m+2)x^2-2my^2$. Below we generalize this observation to other series considered in Theorem 2.

7.4. **Proof of Theorem 3.** As we observed above one can replace the condition $(m+\frac{1}{2})(n+\frac{1}{2})>0$ in the definition of the series from Theorem 2 by any condition of the form $(m+\alpha)(n+\beta)>0$ where $\alpha,\beta\notin\mathbb{Z}$ (due to the vanishing of the similar sum along the lines parallel to the generators on the cone). So we can write this series as follows:

$$F=\sum_{(m,n)\in p\mathbb{Z}\oplus p\mathbb{Z}+\mathbf{c},(m+\frac{1}{4D})(n+\frac{1}{4D})>0}\operatorname{sign}(m+\frac{1}{4D})(-1)^{m_0+n_0}q^{\frac{am^2+2bmn+cn^2}{2}},$$

where $\mathbf{c}=(r+\frac{pc(bs_2-as_1)}{2D},r+\frac{pa(bs_1-cs_2)}{2D})$, $(m_0,n_0)\in\mathbb{Z}^2$ is defined by $(m,n)=p(m_0,n_0)+\mathbf{c}$. Let us take $m'=m/p$ and $n'=(n+\frac{b}{c}m)/p$ as the new summation variables. Note that if $(m,n)=p(m_0,n_0)+\mathbf{c}$ then

$$(m',n')=(m_0,n_0+\frac{b}{c}m_0)+(\frac{r}{p}+\frac{acs}{2D},r\frac{b/c+1}{p}+\frac{s_2}{2}),$$

where $s=-s_1+\frac{b}{a}s_2$. Thus, (m',n') runs through the coset $\mathbb{Z}^2+(\frac{r}{p}+\frac{acs}{2D},\frac{1}{2})$ intersected with the cone $(m'+\varepsilon)(n'-\frac{b}{c}m'+\varepsilon)>0$, where $\varepsilon>0$ is sufficiently small. Therefore, we can write

$$\pm F=\sum_{(m,n)\in\mathbb{Z}^2+(\frac{r}{p}+\frac{acs}{2D},\frac{1}{2}),(m+\varepsilon)(n-\frac{b}{c}m+\varepsilon)>0}\operatorname{sign}(m+\varepsilon)(-1)^{n_0+(\frac{b}{c}+1)m_0}q^{p^2(cn^2-\frac{D}{c}m^2)},$$

where $(m,n)=(m_0,n_0)+(\frac{r}{p}+\frac{acs}{2D},\frac{1}{2})$ (the sign in front of F depends on the parity of $r(b/c+1)/p+(s_2-1)/2$). Now the lattice Λ defined in the formulation of the theorem comes into play. Namely, splitting the above sum in two pieces according to the parity of $n_0+(\frac{b}{c}+1)m_0$ we obtain

$$\pm F=\sum_{(m,n)\in(\Lambda+(\frac{r}{p}+\frac{acs}{2D},\frac{1}{2}))\cap C}\operatorname{sign}(m+\varepsilon)q^{\frac{p^2}{2}Q(m,n)}-$$
$$\sum_{(m,n)\in(\Lambda+(\frac{r}{p}+\frac{acs}{2D},-\frac{1}{2}))\cap C}\operatorname{sign}(m+\varepsilon)q^{\frac{p^2}{2}Q(m,n)},$$

where C is the cone $(m+\varepsilon)(n-\frac{b}{c}m+\epsilon)>0$. It is convenient to identify the lattice Λ with a $\mathbb{Z}$-submodule in the field $K=\mathbb{Q}(\sqrt{D})$ by associating to $(m,n)\in\Lambda$ the element $cn+m\sqrt{D}$. Then we have

$$\frac{1}{2}Q(m,n)=\frac{1}{c}\operatorname{Nm}(cn+m\sqrt{D}).$$

For two non-zero elements $k_1,k_2\in K$ let us denote by $\langle k_1,k_2\rangle=\mathbb{Q}_{>0}k_1+\mathbb{Q}_{>0}k_2$, $[k_1,k_2]=\mathbb{Q}_{\geq 0}k_1+\mathbb{Q}_{\geq 0}k_2$, $\langle k_1,k_2]=\mathbb{Q}_{\geq 0}k_1+\mathbb{Q}_{>0}k_2$. The intersection of a Λ-coset

with the cone C is equal to its intersection with the set $[1, b+\sqrt{D}] \cup \langle -1, -b-\sqrt{D}\rangle$. Making the change of variables $(m,n) \mapsto (m,-n)$ in the second sum of the above expression for $\pm F$ we can write:

$$\pm F = \sum_{(m,n)\in(\Lambda+\mathbf{c})\cap([1,b+\sqrt{D}]\cup\langle -1,-b-\sqrt{D}\rangle)} \operatorname{sign}(m+\varepsilon) q^{\frac{p^2}{2}Q(m,n)} - \sum_{(m,n)\in(\Lambda+\mathbf{c})\cap([-1,-b+\sqrt{D}]\cup\langle 1,b-\sqrt{D}\rangle)} \operatorname{sign}(m+\varepsilon) q^{\frac{p^2}{2}Q(m,n)},$$

where $\mathbf{c} = (\frac{r}{p} + \frac{acs}{2D}, \frac{1}{2})$. Since $\Lambda + \mathbf{c}$ doesn't contain zero, the obtained series is equal to

$$\sum_{(m,n)\in(\Lambda+\mathbf{c})\cap(\langle b-\sqrt{D},b+\sqrt{D}]\cup\langle -b-\sqrt{D},-b+\sqrt{D}])} \operatorname{sign}(cn + m\sqrt{D}) q^{\frac{p^2}{2}Q(m,n)}.$$

Let us consider the totally positive element $\epsilon = \frac{b+\sqrt{D}}{b-\sqrt{D}} \in K$. Since $\mathrm{Nm}(\epsilon) = 1$ the multiplication by ϵ preserves the quadratic form Q on Λ. The direct computation shows that the multiplication by ϵ preserves also $\Lambda + \mathbf{c}$, so we have

$$\pm F = \sum_{(m,n)\in(\Lambda+\mathbf{c})\cap(Q>0)/G_\epsilon} \operatorname{sign}(cn + m\sqrt{D}) q^{\frac{p^2}{2}Q(m,n)},$$

where $G_\epsilon \subset K^*$ is the subgroup generated by ϵ. Let G be the subgroup of the group $\ker(\mathrm{Nm} : K^* \to \mathbb{Q}^*)$ consisting of elements k such that k is totally positive and $k(\mathcal{L} + \mathbf{c}) = \mathcal{L} + \mathbf{c}$. Then G_ϵ is a subgroup of finite index in G, so we have

$$\pm F = |G/G_\epsilon| \cdot \sum_{(m,n)\in(\Lambda+\mathbf{c})\cap(Q>0)/G} \operatorname{sign}(cn+m\sqrt{D}) q^{\frac{p^2}{2}Q(m,n)} = |G/G_\epsilon| \cdot \Theta^H_{\Lambda,Q;\mathbf{c}}(p^2\tau).$$

$\square$

7.5. **Determinantal Jacobi forms.** We are going to compute explicitly the map $MP^{\det}_{s_1,s_2}$ (see section 6.3) in the setup of section 7.1. Note that we can trivialize the 1-dimensional vector space M using the canonical bases in the relevant vector spaces. First, let us compute the map

$$e_{s_1,s_2} : \bigwedge^{d_0-1} \mathrm{Hom}^1(L_1, L_2)^* \to \mathrm{Hom}^1(L_1, L_2).$$

For every subset $S \subset \mathbb{Z}/d\mathbb{Z}$ of cardinality $d - d_0 + 1$ let us denote by e^S_{12} the element in $\bigwedge^{d_0-1} \mathrm{Hom}^1(L_1, L_2)^*$ which is induced by the projection

$$\mathrm{Hom}^1(L_1, L_2) \to \mathbb{C}^{\oplus d_0-1} : \sum_k y_k e_{12}(0,k) \mapsto (y_k, k \notin S).$$

Then

$$e_{s_1,s_2}(e^S_{12}) = \sum_{k\in S} c_{k,S} e_{12}(0,k)$$

where $(c_{k,S}, k \in S)$ is the sequence of $(d-d_0)\times(d-d_0)$-minors (with signs) of the $(d-d_0)\times(d-d_0+1)$-matrix R obtained by putting together the $(d-d_1)\times(d-d_0+1)$-matrix $R_1 = (A_{ik}; i \in \mathbb{Z}/(d-d_1)\mathbb{Z}, k \in S)$ and the $(d-d_2)\times(d-d_0+1)$-matrix $R_2 = (B_{jk}; j \in \mathbb{Z}/(d-d_2)\mathbb{Z}, k \in S)$, where A_{ik} is zero unless $k \equiv i(d, d-d_1)$ in which case

$$A_{ik} = \theta_{I_1, -\frac{k}{d}+n_1(k,i)}(p_1x_1, p_1\tau),$$

where $n_1(k,i) \in \mathbb{Z}/I_1$ is characterized by the congruence $dn_1(k,i) \equiv k - i(d-d_1)$; similarly B_{jk} is zero unless $k \equiv j(d, d-d_2)$ in which case

$$B_{jk} = \theta_{I_2, \frac{k}{d}+n_2(k,j)}(p_2x_2, p_2\tau),$$

where $n_2(k,j) \in \mathbb{Z}/I_2$ is characterized by the congruence $dn_2(k,j) \equiv j - k(d-d_2)$.

Now we have

$$F_{l,S} = \langle e_{03}^*(0,l), MP_{s_1,s_2}^{\det}(e_{12}^S)\rangle = \sum_{k\in S} C_{l,k}c_{k,S},$$

where $C_{l,k}$ are defined by (7.3). In other words, $F_{l,S}$ is equal to the determinant of the $(d-d_0+1)\times(d-d_0+1)$-matrix obtained by putting together R_1, R_2 and the row $(C_{l,k}, k \in S)$ of length $d-d_0+1$.

Let us recall the definition of Jacobi forms from [6]. Let Λ be a lattice, $\mathbf{n}\cdot\mathbf{n}'$ be a symmetric bilinear form on Λ with values in $\mathbb{Z}$, $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a congruenz-subgroup. Let us denote $Q(\mathbf{n}) = \mathbf{n}\cdot\mathbf{n}$ (this corresponds to $2Q$ in the notation of [6]). Then a meromorphic function $f(\mathbf{z},\tau)$ on $\Lambda_{\mathbb{C}} \times \mathfrak{H}$ is called a (meromorphic) Jacobi form of weight k with respect to (Λ, Q, Γ) if the following equations hold:

$$f(\mathbf{z}+\mathbf{v}\tau+\mathbf{w},\tau) = (-1)^{\mathbf{v}\cdot\mathbf{w}}\exp(-\pi i\tau Q(\mathbf{v}) - 2\pi i\mathbf{v}\cdot\mathbf{z})f(\mathbf{z},\tau),$$

$$f(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k\exp(\pi i\frac{cQ(\mathbf{z})}{c\tau+d})f(z,\tau),$$

for every $\mathbf{v},\mathbf{w}\in\Lambda$, $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$. Similar to the case of modular forms one can extend the above definition to half-integer weights k.

Theorem 9. *The function $F_{l,S}$ is a Jacobi form of weight $(d-d_0)/2+1$ with respect to some sublattice $\Lambda' \subset \Lambda^+$, some congruenz-subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$, and the quadratic form $Q+Q_0$, where*

$$\Lambda^+ = \{\mathbf{n} = (n_0,n_1,n_2,n_3) \in \mathbb{Z}^4 : \sum_i n_i = 0, d_1n_1 + (d_0-d_2)n_2 + d_0n_3 = 0\},$$

$$Q(\mathbf{n}) = -d_1n_0n_1 - d_2n_2n_3,$$
$$Q_0(\mathbf{n}) = d(d_1n_1^2 + d_2n_2^2).$$

Proof. We have the system of linear equations

$$\sum_l D_{k,l}F_{l,S} = c_{k,S}$$

determining $F_{l,S}$. Using the functional equation for theta function we derive that $D_{k,l}$ are Jacobi forms of weight -1 with respect to some congruenz-subgroup of $\mathrm{SL}(2,\mathbb{Z})$, some sublattice of Λ^+ and the quadratic form $-Q$. On the other hand, $c_{k,S}$ are Jacobi forms of weight $(d-d_0)/2$ with respect to the quadratic form Q_0. Hence, $F_{l,S}$ are Jacobi forms of weight $(d-d_0)/2+1$ with respect to $Q+Q_0$. □

Remark. It is easy to see that the form $Q+Q_0$ can be written as follows:

$$(Q+Q_0)(\mathbf{n}) = \frac{d_1d_2}{d_0}(x_1+x_2)^2 + \frac{d(d_0-1)}{d_0}(d_1x_1^2+d_2x_2^2).$$

In particular, it is always positive-definite for $d_0 > 1$. In the case $d_0 = 1$ this form is degenerate which means that the corresponding functions $F_{l,S}$ have form $f(x_1+x_2,\tau)$, where $f(z,\tau)$ is a Jacobi form.

For example, for $d_1 = d_2 = 2$, $d_0 = 1$ we have the unique choice of l and S. Let us take $z_1 = x_0, z_2 = x_3$ as coordinates in $\Lambda^+_{\mathbb{C}}$. Then the corresponding function is equal (up to a sign) to the following deteminant:

$$F(x_1, x_2; \tau) = \det \begin{pmatrix} \Theta_0(x_1, x_2; \tau) & \Theta_{-1}(x_1, x_2; \tau) & \Theta_1(x_1, x_2; \tau) \\ \theta_0(6x_1, 6\tau) & \theta_{-\frac{1}{3}}(6x_1, 6\tau) & \theta_{\frac{1}{3}}(6x_1, 6\tau) \\ \theta_0(6x_2, 6\tau) & \theta_{\frac{1}{3}}(6x_2, 6\tau) & \theta_{-\frac{1}{3}}(6x_2, 6\tau) \end{pmatrix}$$

where we denoted $\theta_r = \theta_{\mathbb{Z},r}$,

$$\Theta_c = \sum_{m-n\equiv c(3),(m+\alpha(2x_2-x_1))(n+\alpha(2x_1-x_2))>0} \operatorname{sign}(m + \alpha(2x_2 - x_1)) \times$$
$$\exp(\pi i\tau\frac{2}{3}(m^2 + 4mn + n^2) + 4\pi i(mx_1 + nx_2)).$$

The function F is a Jacobi form in $(x_1 + x_2, \tau)$ of weight 2 and index 2. Using the equation $D_{0,0}F = c_0$ and the addition formulas for theta functions we derive the following identity:

$$F(x_1, x_2; \tau) = \theta_{\frac{1}{2}}(2(x_1 + x_2) + \frac{1}{2}, 2\tau)^2 \cdot \frac{\eta^3(2\tau)\sum_{n\in\mathbb{Z}}\chi_3(n)q^{\frac{(4n+3)^2}{24}}}{\theta(\frac{1}{2}, 2\tau)\theta_{\frac{1}{4}}(0, 4\tau)}$$

where $\chi_3(n)$ is the non-trivial Dirichlet character modulo 3.

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